Journal of Approximation Theory **104**, 45–76 (2000)
doi:10.1006/jath.1999.3438, available online at http://www.idealibrary.com on IDEAL®

Knot Insertion and Totally Positive Systems

E. Mainar and J. M. Peña[1]

Departamento de Matemática Aplicada, Universidad de Zaragoza, 50009 Zaragoza, Spain

Communicated by Günther Nürnberger

Received January 26, 1999; accepted in revised form October 22, 1999

It is analyzed when a given space of functions admits a knot insertion algorithm. Techniques related to total positivity will provide the construction of such knot insertion algorithms in a general space admitting shape preserving representations. Many examples are included. © 2000 Academic Press

Key Words: knot insertion; splines; total positivity; generalized splines; Tchebycheffian splines; shape preservation.

1. INTRODUCTION

Total positivity and its associated variation diminishing property are closely related to some fundamental questions in approximation theory (see [13]). For instance, it is well known that B-splines collocation matrices are totally positive, and some recent papers [6, 7] have shown the strong relationship between Tchebycheff and weak Tchebycheff systems and totally positive systems. Totally positive bases have also been found for the generalized spline spaces of [12, 20], for the L-splines space (see [15]), and for the Tchebycheffian splines space (see [19, Chap. 9]). The class of Tchebycheffian splines contains polynomial, exponential, and hyperbolic splines, among other many classes of splines. Algorithms for Tchebycheffian splines have been introduced following different approaches: by using recurrence relations (see [14, 8]) and, more recently, by using techniques related with blossoming (see [18, 17]), by generalized de Boor–Fix dual functionals (see [1]), or by integration (see [2, 3]). In this paper we analyze the role of total positivity in order to construct a knot insertion procedure in a given space of functions. We can apply our results to the generalized splines of [12], which contain the mentioned Tchebycheffian splines and trigonometric splines.

In computer aided geometric design, shape preserving representations using control polygons are those representations associated with normalized

[1] This research has been partially supported by Research Grant DGICYT PB96-0730.

0021-9045/00 $35.00
Copyright © 2000 by Academic Press
All rights of reproduction in any form reserved.

totally positive bases (see [11, 4]). The basis with optimal shape preserving properties has been called the normalized B-basis in [5], where its existence and uniqueness are proved. Examples of B-bases are the Bernstein basis and the B-spline basis. When working with a space $\mathcal{P}$ of polynomial splines, given a curve expressed in terms of the B-spline basis of $\mathcal{P}$, knot insertion allows us to express the curve in terms of other B-spline bases of spaces containing $\mathcal{P}$, providing more flexibility for the interactive design of the curve. In this paper we generalize the knot insertion procedure by replacing the space $\mathcal{P}$ by a space $\mathcal{U}$ with shape preserving representations and the B-spline bases by the corresponding normalized B-bases of spaces containing $\mathcal{U}$.

In [16] it is derived a corner cutting algorithm associated to any normalized B-basis (called B-algorithm) which is an evaluation algorithm and satisfies subdivision properties. For instance, de Casteljau algorithm is a B-algorithm. In [16] it is also proved that the control polygons obtained after iterating B-algorithms converge to the curve. B-algorithms lead to normalized B-bases of spaces with greater dimension. In this paper we study when a B-algorithm also provides a knot insertion algorithm, so that the concept of B-algorithm, which can be applied to any space admitting shape preserving representations, unifies three important types of algorithms in computer aided geometric design: evaluation, subdivision, and knot insertion algorithms.

In Section 2 we give some basic results and introduce the concepts of k-admissible parameter and symmetric B-algorithm, which will be key tools in this paper. In Section 3 we study the structure of the matrices associated to B-algorithms. In Section 4 we introduce the basic definitions in order to generalize the concept of knot insertion in a given space with shape preserving representations. In Section 5 we show, for a given B-algorithm, that the abstract property of symmetry is equivalent to the fact that it provides a knot insertion algorithm. In Theorem 6.1 we prove that B-algorithms corresponding to admissible parameters are always symmetric and therefore provide knot insertion algorithms. Although there are B-algorithms which are not symmetric (as shown in Example 7.1), we include in Section 7 many examples of spaces such that the B-algorithms are symmetric and so provide knot insertion algorithms.

2. AUXILIARY RESULTS

Let $\mathcal{U}$ be a vector space of real functions defined on $I \subseteq \mathbf{R}$ and let $(u_0, ..., u_n)$ be a basis of $\mathcal{U}$. Given $D \subseteq I$, and $u \in \mathcal{U}$ we shall denote by $\mathcal{U}|_D$ the space $\{u(t) \mid u \in \mathcal{U}, t \in D\}$. If a sequence $P_0, ..., P_n$ of points in $\mathbf{R}^k$ is given then we define a curve $\gamma(t) = \sum_{i=0}^{n} P_i u_i(t)$, $t \in I$. The points $P_0, ..., P_n$

are called *control points* and the polygon $P_0 \cdots P_n$ with vertices $P_0, ..., P_n$ is called the *control polygon* of γ. The system of functions $(u_0, ..., u_n)$ is *normalized* if $\sum_{i=0}^n u_i(t) = 1$, $\forall t \in I$. The *collocation matrix* of $(u_0(t), ..., u_n(t))$ at $t_0 < \cdots < t_m$ in I is given by

$$M \begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_m \end{pmatrix} := (u_j(t_i))_{i=0, ..., m;\ j=0, ..., n}. \tag{2.1}$$

Clearly, $(u_0, ..., u_n)$ is normalized and formed by nonnegative functions if and only if all its collocation matrices are stochastic (that is, nonnegative and such that the sum of each row is one). We shall use the following matricial notation. Given an $m \times n$ matrix A and $\alpha = (\alpha_1, ..., \alpha_k)$, $\beta = (\beta_1, ..., \beta_l)$ with $1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant m$, $1 \leqslant \beta_1 < \cdots < \beta_l \leqslant n$, we denote by $A[\alpha \mid \beta]$ the $k \times l$ submatrix of A containing rows α and columns β, and $A[\alpha] := A[\alpha \mid \alpha]$. The identity matrix of order m will be denoted by I_m. A matrix is *totally positive* if all its minors are nonnegative and a system of functions is totally positive when all its collocation matrices (2.1) are totally positive. It is well known (cf. [11]) that shape preserving representations of curves by means of control polygons must be associated with normalized totally positive bases.

The following result for totally positive bases is a consequence of Lemma 2.1 and Proposition 4.1 of [5]:

PROPOSITION 2.1. *Let $(u_0, ..., u_n)$ be a totally positive basis of a vector space of functions defined on an interval $I \subseteq \mathbf{R}$ and let $I_i := \{t \in I \mid u_i(t) \neq 0\}$. Then the following properties hold*:

(i) *The function $u_j(t)/u_i(t)$ defined on I_i is monotonic increasing (resp., decreasing) for all $j > i$ (resp., $j < i$).*

(ii) *If $(u_0, ..., u_n)$ is also normalized then, for each $i = 0, ..., n$, I_i is an interval whose infimum (resp., supremum) $\alpha_i \in \mathbf{R} \cup \{-\infty\}$ (resp., $\beta_i \in \mathbf{R} \cup \{+\infty\}$) satisfies*

$$\alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n \qquad (\textit{resp.}, \beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_n). \tag{2.2}$$

Normalized totally positive bases with optimal shape preserving properties among all normalized totally positive bases of the space have been called normalized B-bases: the curve γ better imitates the shape of the control polygon with respect to the normalized B-basis than the shape of the control polygon with respect to any other normalized totally positive basis (see [4, 5]). Examples of B-bases are the Bernstein basis in the case of the space $\mathbf{P}^k([a, b])$ of polynomials of degree less than or equal to n on an interval $[a, b]$ and the B-spline basis in the case of the corresponding polynomial spline space.

In [5] the concept of B-basis was defined in terms of two sequences of vector subspaces of $\mathcal{U}$,

$$\mathcal{U}=\mathcal{L}^0(\mathcal{U})\supset\mathcal{L}^1(\mathcal{U})\supset\cdots\supset\mathcal{L}^n(\mathcal{U}),\quad \mathcal{U}=\mathcal{R}^0(\mathcal{U})\supset\mathcal{R}^1(\mathcal{U})\supset\cdots\supset\mathcal{R}^n(\mathcal{U}) \tag{2.3}$$

(see [5, pp. 641–642] for the definition and construction of these subspaces).

DEFINITION 2.2. A totally positive basis $(b_0, ..., b_n)$ is a *B-basis* if $b_i \in L^i(\mathcal{U}) \cap R^{n-i}(\mathcal{U})$, $i=0, ..., n$.

In Proposition 3.12 of [5] it was given the following characterization of a B-basis:

PROPOSITION 2.3. *Let* $(u_0, ..., u_n)$ *be a totally positive basis of a space* $\mathcal{U}$ *defined on* $I\subseteq \mathbf{R}$. *Then* $(u_0, ..., u_n)$ *is a B-basis if and only if the following conditions hold*

$$\inf\left\{\frac{u_i(t)}{u_j(t)}\,\middle|\, t\in I,\, u_j(t)\neq 0\right\}=0, \tag{2.4}$$

for all $i\neq j$.

Taking into account formulae (3.1) and (3.2) of [5] together with Propositions 3.3 and 3.5 of [5] we can derive the following result:

PROPOSITION 2.4. *Let* $(b_0, ..., b_n)$ *be a B-basis of* $\mathcal{U}$. *Then* $(b_i, ..., b_n)$ *is a basis of* $L^i(\mathcal{U})$ *and* $(b_0, ..., b_i)$ *is a basis of* $R^{n-i}(\mathcal{U})$.

Existence of B-bases and normalized B-bases follows from Remark 3.8 and Theorem 4.2(i) of [5], respectively:

PROPOSITION 2.5. *If a vector space of functions has a totally positive* (*resp., normalized totally positive*) *basis then it has a B-basis* (*resp., a unique normalized B-basis*).

Associated to any normalized B-basis we construct in [16] a corner cutting algorithm (which we call a B-algorithm) satisfying evaluation and subdivision properties. On the other hand, the B-algorithm transforms the control polygon of a curve with respect to a normalized B-basis into the control polygon with respect to a normalized B-basis of a space with greater dimension. In this paper we shall analyze the relationship between B-algorithms and knot insertion algorithms. From now on, we assume that $I\subseteq\mathbf{R}$ is an interval and that given $t_0\in \mathrm{Int}(I)$ (the interior of I), $I' := (-\infty, t_0]\cap I$ and $I'' := (t_0, \infty)\cap I$. Let $(u_0, ..., u_n)$ be a totally positive basis of a vector space $\mathcal{U}$ of functions defined on I. Since $u_0(t), ..., u_n(t)$, $t \in I'$ (resp., $t \in I''$), form a totally positive system, by Proposition 2.5 they generate a space which has a B-basis $(\bar{v}_0, ..., \bar{v}_r)$ (resp., $(\bar{w}_0, ..., \bar{w}_s)$). Let us define

$$v_i(t) := \begin{cases} \bar{v}_i(t) & \text{if} \quad t \in I' \\ 0 & \text{if} \quad t \in I'' \end{cases}, \quad i = 0, ..., r \tag{2.5}$$

and

$$w_i(t) := \begin{cases} 0 & \text{if} \quad t \in I' \\ \bar{w}_i(t) & \text{if} \quad t \in I'' \end{cases}, \quad i = 0, ..., s. \tag{2.6}$$

These functions generate a vector space of functions

$$\mathcal{V} := \text{span}\{v_0(t), v_1(t), ..., v_r(t), w_0(t), w_1(t), ..., w_s(t)\}, \qquad t \in I. \tag{2.7}$$

By construction,

$$\mathcal{U}|_{I'} = \mathcal{V}|_{I'}, \qquad \mathcal{U}|_{I''} = \mathcal{V}|_{I''}. \tag{2.8}$$

It is easy to see that

$$L^i(\mathcal{U})|_{I'} \subseteq L^i(\mathcal{U}|_{I'}), \qquad R^{n-i}(\mathcal{U})|_{I''} \subseteq R^{n-i}(\mathcal{U}|_{I''}), \qquad i = 0, ..., n. \tag{2.9}$$

In Proposition 3.1 of [16] it was shown that $\mathcal{U} \subseteq \mathcal{V}$, that $(v_0, ..., v_r, w_0, ..., w_s)$ is a B-basis of $\mathcal{V}$ and that any B-basis of $\mathcal{V}$ has this form. In Theorem 3.3 of the same paper it was shown that if $(u_0, ..., u_n)$ is a normalized B-basis of a space $\mathcal{U}$ then there exists a normalized B-basis $(v_0, ..., v_r, w_0, ..., w_s)$ of the vector space $\mathcal{V}$ and that the $(r+s+2) \times (n+1)$ matrix M such that

$$(u_0, ..., u_n) = (v_0, ..., v_r, w_0, ..., w_s)\, M \tag{2.10}$$

satisfies the following properties:

(i) $L := M[1, ..., r+1]$ is a lower triangular nonsingular stochastic totally positive matrix. The elements of $M[1, ..., r+1 \,|\, r+2, ..., n+1]$ are zeros.

(ii) $U := M[r+2, ..., r+s+2 \,|\, n-s+1, ..., n+1]$ is an upper triangular nonsingular stochastic totally positive matrix. The elements of $M[r+2, ..., r+s+2 \,|\, 1, ..., n-s]$ are zeros.

From (2.10), (2.5), and (2.6) we see that

$$\begin{aligned} u_i(t) &= 0, \qquad \forall t \in I', \quad i = r+1, ..., n; \\ u_i(t) &= 0, \qquad \forall t \in I'', \quad i = 0, ..., n-s-1. \end{aligned} \tag{2.11}$$

We also have that

$$(u_0(t), ..., u_r(t)) = (v_0(t), ..., v_r(t))\, L, \qquad \forall t \in I', \tag{2.12}$$

and

$$(u_{n-s}(t), ..., u_n(t)) = (w_0(t), ..., w_s(t))\, U, \qquad \forall t \in I''. \tag{2.13}$$

Remark 2.6. If $(u_0, ..., u_n)$ is a normalized B-basis of a space $\mathcal{U}$ of functions continuous at t_0 we know from (2.11) that the basis functions which do not vanish at t_0 belong to the set $\{u_{n-s}, ..., u_r\}$. Then taking into account that $\dim(\mathcal{U}|_{I'}) = r+1$, $\dim(\mathcal{U}|_{I''}) = s+1$, (2.11) and Proposition 2.1(ii), we can deduce that $u_i(t_0) \neq 0$ if and only if $i \in \{n-s, ..., r\}$ and so the number of functions of the basis nonvanishing at t_0 is $k = r+s-n+1$.

As it is recalled in Section 2 of [16], since L and U (of (2.12) and (2.13), respectively) are nonsingular stochastic totally positive triangular matrices, they can be factorized (in a unique way) as

$$L = L_{r-1} L_{r-2} \cdots L_0,$$

$$L_l = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & l_l^{(l)} & 1-l_l^{(l)} & & \\ & & & & \ddots & \ddots & \\ & & & & & l_{r-1}^{(l)} & 1-l_{r-1}^{(l)} \end{pmatrix} \tag{2.14}$$

with $0 \leqslant l_j^{(l)} < 1$ (for $l = 0, ..., r-1$, $j = l, ..., r-1$) and

$$U = U_{s-1} U_{s-2} \cdots U_0,$$

$$U_l = \begin{pmatrix} u_l^{(l)} & 1-u_l^{(l)} & & & & & \\ & \ddots & \ddots & & & & \\ & & u_{s-1}^{(l)} & 1-u_{s-1}^{(l)} & & & \\ & & & 1 & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix} \tag{2.15}$$

with $0 < u_j^{(l)} \leqslant 1$ (for $l = 0, ..., s-1$, $j = l, ..., s-1$).

These factorizations have an interpretation in terms of corner cutting algorithms. An *elementary corner cutting* is a transformation which maps any polygon $P_0 \cdots P_n$ into another polygon $B_0 \cdots B_n$ defined by one of the following ways

$$\begin{aligned} B_j &= P_j, \qquad j \neq i, \\ B_i &= (1-\lambda)\, P_i + \lambda P_{i+1}, \end{aligned} \qquad \text{for some} \quad i \in \{0, ..., n-1\}, \quad 0 \leqslant \lambda < 1$$

or

$$\begin{aligned} B_j &= P_j, \qquad j \neq i, \\ B_i &= (1-\lambda)\, P_i + \lambda P_{i-1}, \end{aligned} \qquad \text{for some} \quad i \in \{1, ..., n\}, \quad 0 \leqslant \lambda < 1.$$

An elementary corner cutting is defined by a one-banded, nonsingular, totally positive and stochastic matrix. A *corner cutting algorithm* is any composition of elementary corner cuttings. A corner cutting algorithm is described by a matrix which is nonsingular, totally positive and stochastic, as a product of the previous ones.

DEFINITION 2.7. Let M, L, U as in (2.10), (2.12), and (2.13). The *B-algorithm* associated to t_0 is the corner cutting algorithm corresponding to the factorizations (2.14) and (2.15) of L and U. It transforms the control polygon of a curve γ with respect to the normalized B-basis of $\mathscr{U}$ into the control polygon with respect to the normalized B-basis of $\mathscr{V}$. The *left B-algorithm* is the corner cutting algorithm corresponding to (2.14) and the *right B-algorithm* is the corner cutting algorithm corresponding to (2.15).

The following definition will play a crucial role in this paper.

DEFINITION 2.8. Let L and U be the matrices associated to the left and right (respectively) B-algorithm. We say that the B-algorithm is *symmetric* if there exists $h \leqslant \min\{r, s\}$ such that the factorizations (2.14) and (2.15) satisfy $L = L_{r-1} \cdots L_{r-h}$ (i.e., $L_{r-h-1}, ..., L_0 = I_{r+1}$), $U = U_{s-1} \cdots U_{s-h}$ (i.e., $U_{s-h-1}, ..., U_0 = I_{s+1}$) and $l^{(r-j)}_{r-m} = u^{(s-j)}_{s-m}$ for all $j = h, ..., 1$ and $m = j, ..., 1$.

We say that a space $\mathscr{U}$ of functions is $\mathscr{C}^j$ $(j \geqslant 0)$ if every $u \in \mathscr{U}$ is j-times continuously differentiable. We can also use $\mathscr{C}$ instead of $\mathscr{C}^0$.

By Remark 2.6 we know that if $(u_0, ..., u_n)$ is a normalized B-basis of functions continuous at t_0 then the number of basis functions nonvanishing at t_0 is $k = r + s + 1 - n (> 0)$. This allows us to give the following definition.

DEFINITION 2.9. Let $\mathscr{U}$ be a vector space of functions defined on I with a normalized B-basis $(u_0, ..., u_n)$, let $t_0 \in \operatorname{Int}(I)$ and let k be the number of basis functions which do not vanish at t_0. We say that t_0 is a *k-admissible* parameter in $\mathscr{U}$ if $(u_0, \ldots, u_n)$ satisfies the following properties:

(A) There exists $\varepsilon > 0$ such that $\mathscr{U}$ is $\mathscr{C}^{k-1}$ in $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq I$.

(B) $\det(u_i^{(j)}(t_0))_{i = n-s, \ldots, r;\, j = 0, \ldots, k-1} \neq 0$.

Properties (A) and (B) of Definition 2.9 are clearly satisfied by all parameters of the interior of the interval of definition of polynomial B-splines. More examples with generalized B-splines will appear in Section 7.

Remark 2.10. Let us observe that, by (2.11) and property (A) of Definition 2.9, for each $i = 0, \ldots, n-s-1$ or $i = r+1, \ldots, n$, u_i has null derivatives at t_0 up to order $k-1$. Then one can easily deduce that with (A), property (B) holds if and only if there exist $f_i \in \mathscr{U}$, $i = 0, \ldots, k-1$ satisfying $f_i^{(j)}(t_0) = 0$ for all $j < i$ and $f_i^{(i)}(t_0) \neq 0$.

PROPOSITION 2.11. *Let $\mathscr{U}$ be a vector space of functions defined on I with a normalized totally positive basis. If t_0 is a k-admissible parameter in $\mathscr{U}$ then $R^i(\mathscr{U}|_{I'}) = \mathscr{Z}_i|_{I'}$ and $L^i(\mathscr{U}|_{I''}) = \mathscr{Z}_i|_{I''}$ where $\mathscr{Z}_0 := \mathscr{U}$ and $\mathscr{Z}_i := \{u \mid u \in \mathscr{U}, u^{(j)}(t_0) = 0, \forall j \leqslant i-1\}$ for each $1 \leqslant i \leqslant k$.*

Proof. By property (A) of Definition 2.9, the spaces $\mathscr{Z}_i$ are well defined. Clearly, $\mathscr{Z}_0 \supseteq \mathscr{Z}_1 \supseteq \cdots \supseteq \mathscr{Z}_k$ and by Remark 2.10 there exists $f_i \in \mathscr{Z}_i \backslash \mathscr{Z}_{i+1}$, $i = 0, \ldots, k-1$. Let $(v_0, \ldots, v_r)$ be a B-basis of $\mathscr{U}|_{I'}$. By Proposition 2.4, $\dim(R^i(\mathscr{U}|_{I'})) = r+1-i$ and from Propositions 2.1(i) and 2.3 one can deduce that $R^i(\mathscr{U}|_{I'}) \subseteq \mathscr{Z}_i|_{I'}$. Let $\tilde{f}_i := f_i|_{I'}$. It can be easily checked that $(v_0, \ldots, v_{r-k}, \tilde{f}_{k-1}, \tilde{f}_{k-2} \ldots, \tilde{f}_i)$ $(i = k-1, \ldots, 0)$ is a linearly independent system and then $\mathscr{F}_i := \operatorname{span}\{v_0, \ldots, v_{r-k}, \tilde{f}_{k-1}, \ldots, \tilde{f}_i\}$ is an $(r-i+1)$-dimensional space. We can now write

$$\mathscr{U}|_{I'} = \mathscr{Z}_0|_{I'} \supseteq \mathscr{F}_0 \supset \mathscr{Z}_1|_{I'} \supseteq \mathscr{F}_1 \supset \cdots \supset \mathscr{Z}_{k-1}|_{I'} \supseteq \mathscr{F}_{k-1} \supset \mathscr{Z}_k|_{I'} \supseteq R^k(\mathscr{U}|_{I'}). \tag{2.16}$$

Taking into account that $\dim(\mathscr{U}|_{I'}) = r+1$, that $\dim(\mathscr{F}_i) = r-i+1$, that $\dim(R^k(\mathscr{U}|_{I'})) = r+1-k$ and (2.16) we deduce that $\mathscr{Z}_i|_{I'} = \mathscr{F}_i$ for each $i = 0, \ldots, k-1$ and $\dim(\mathscr{Z}_i|_{I'}) = r+1-i$ for $i = 0, \ldots, k$. So $R^i(\mathscr{U}|_{I'}) = \mathscr{Z}_i|_{I'}$ $(0 \leqslant i \leqslant k)$. Analogously, it can be proved that $L^i(\mathscr{U}|_{I''}) = \mathscr{Z}_i|_{I''}$ for each $i = 0, \ldots, k$. ∎

Taking into account Propositions 2.4 and 2.11 we can immediately deduce the following result on the B-basis of $\mathscr{U}|_{I'}$ and $\mathscr{U}|_{I''}$ for admissible parameters.

COROLLARY 2.12. *Under the hypotheses of Proposition* 2.11 *if* $(\bar{v}_0, ..., \bar{v}_r)$, $(\bar{w}_0, ..., \bar{w}_s)$ *are B-bases of* $\mathcal{U}|_{I'}$ *and* $\mathcal{U}|_{I''}$, *respectively, then*

$$\bar{v}^{(j)}_{r-i}(t_0)=0,\ \ j<i,\ \ \bar{v}^{(i)}_{r-i}(t_0)\neq 0; \qquad \lim_{t\to t_0^+} \bar{w}^{(j)}_i(t)=0,\ \ j<i,\ \ \lim_{t\to t_0^+} \bar{w}^{(i)}_i(t)\neq 0$$

for all $i=0, ..., k-1$.

The following result will play a crucial role in Theorem 6.1.

LEMMA 2.13. *Let* $(u_0, ..., u_n)$ *be the normalized B-basis of* $\mathcal{U}$. *If* $t_0 \in \mathrm{Int}(I)$ *is a k-admissible parameter in* $\mathcal{U}$ *then the basis functions* $u_{n-s}, ..., u_r$ $(k=r+s+1-n)$ *which do not vanish at* t_0 *are linearly independent on any interval J such that* $t_0 \in J \subseteq I$.

Proof. Let $\alpha_{n-s}u_{n-s}(t)+\cdots+\alpha_r u_r(t)=0$, $\forall t\in J$. Taking into account property (A) of Definition 2.9 we can write for each $j=0, ..., k-1$: $\alpha_{n-s}u^{(j)}_{n-s}(t)+\cdots+\alpha_r u^{(j)}_r(t)=0$, $\forall t\in J\cap(t_0-\varepsilon, t_0+\varepsilon)$. In particular, $\alpha_{n-s}u^{(j)}_{n-s}(t_0)+\cdots+\alpha_r u^{(j)}_r(t_0)=0$, and from property (B) of Definition 2.9 we deduce that $\alpha_{n-s}=\cdots=\alpha_r=0$. ∎

3. MATRICES ASSOCIATED WITH B-ALGORITHMS

In the following result we obtain, the structure of the matrices L and U of (2.12) and (2.13), respectively for admissible parameters.

THEOREM 3.1. *Let* $(u_0, ..., u_n)$ *be the normalized B-basis of a vector space* $\mathcal{U}$ *of functions defined on I. If* $t_0\in\mathrm{Int}(I)$ *is a k-admissible parameter in* $\mathcal{U}$ *then the matrices* $L=(l_{ij})_{0\leq i,j\leq r}$, $U=(u_{ij})_{0\leq i,j\leq s}$ *corresponding to the left and right B-algorithm, respectively are of the form*

$$L=\begin{pmatrix} I_{r-k+1} & 0\\ 0 & \tilde{L}\end{pmatrix}, \qquad U=\begin{pmatrix}\tilde{U} & 0\\ 0 & I_{s-k+1}\end{pmatrix}, \tag{3.1}$$

where $r+1=\dim(\mathcal{U}|_{I'})$ (*resp.*, $s+1=\dim(\mathcal{U}|_{I''})$) *and* $\tilde{L}$ (*resp.*, $\tilde{U}$) *is a* $k\times k$ *nonsingular stochastic lower* (*resp.*, *upper*) *triangular matrix.*

Proof. Let us prove (3.1) for L. Let $(v_0, ..., v_r, w_0, ..., w_s)$ be the normalized B-basis of the space $\mathcal{V}$ defined in (2.7). From (2.12) we have

$$(u_0(t), ..., u_r(t))=(u_0(t), ..., u_{r-k}(t), v_{r-k+1}(t), ..., v_r(t))\,\bar{L}, \qquad t\in I', \tag{3.2}$$

where $\bar{L} = \begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \bar{L}_k \end{pmatrix}$ and $\bar{L}_k := L[r+2-k, ..., r+1]$. It is sufficient to prove that $(u_0(t), ..., u_{r-k}(t), v_{r-k+1}(t), ..., v_r(t))$, $t \in I'$ is a B-basis because then by the uniqueness up to positive scaling of B-bases (see Corollary 3.9(iii) of [5]) there exists a positive nonsingular diagonal matrix D such that $(u_0(t), ..., u_{r-k}(t), v_{r-k+1}(t), ..., v_r(t)) = (v_0(t), ..., v_r(t))\, D$, $t \in I'$ and then from (2.12) and (3.2) we conclude that $L = D\bar{L}$ and the theorem follows for L. Since $(v_0, ..., v_r)$ is the normalized B-basis of $\mathcal{U}|_{I'}$ we know that $v_i \in L^i(\mathcal{U}|_{I'}) \cap R^{r-i}(\mathcal{U}|_{I'})$ for all $i = r-k+1, ..., r$. Since $(u_0, ..., u_n)$ is a normalized B-basis of $\mathcal{U}$ we know that $u_i \in L^i(\mathcal{U})$ for all $i = 0, ..., r-k$ and by (2.9), $u_i \in L^i(\mathcal{U}|_{I'})$. Thus it remains to see that

$$u_i \in R^{r-i}(\mathcal{U}|_{I'}), \qquad \forall i = 0, ..., r-k. \tag{3.3}$$

By Remark 2.6, $k = r+s+1-n$ and then from (2.11) we know that $(u_0, ..., u_n)$ is a B-basis of $\mathcal{U}$ such that $u_0(t) = \cdots = u_{r-k}(t) = 0$, $\forall t \geqslant t_0$. By property (A) of Definition 2.9, $u_0^{(j)}(t_0) = \cdots = u_{r-k}^{(j)}(t_0) = 0$ for each $j = 0, ..., k-1$ and then from Proposition 2.11 we deduce that $u_j \in R^k(\mathcal{U}|_{I'})$ for all $j = 0, ..., r-k$. By Proposition 2.4, $\dim(R^k(\mathcal{U}|_{I'})) = r-k+1$ and $(v_0, ..., v_{r-k-1}, v_{r-k})$ is a totally positive basis of $R^k(\mathcal{U}|_{I'})$. Since $u_{r-k} \in R^k(\mathcal{U}|_{I'})$ then from Corollary 3.9(i) of [5] we have that $u_{r-k} = \alpha v_{r-k}$ $(\alpha > 0)$ and so the system $(v_0, ..., v_{r-k-1}, u_{r-k})$ is a B-basis of $R^k(\mathcal{U}|_{I'})$. From Proposition 2.3 we can write now that

$$0 = \inf \left\{ \frac{u_j(t)}{u_{r-k}(t)} \,\middle|\, t \in I, u_{r-k}(t) \neq 0 \right\} = \inf \left\{ \frac{u_j(t)}{u_{r-k}(t)} \,\middle|\, t \in I', u_{r-k}(t) \neq 0 \right\}, \tag{3.4}$$

for each $j = 0, ..., r-k-1$. Therefore, by Proposition 3.5 of [5], $u_j \in R(R^k(\mathcal{U}|_{I'})) = R^{k+1}(\mathcal{U}|_{I'})$ for each $j = 0, ..., r-k-1$. Iterating the previous argument we can get (3.3). A similar proof of (3.1) can be derived for U, by considering the normalized B-basis of $\mathcal{U}|_{I''}$ $(w_0, ..., w_s)$ and showing that the system $(w_0(t), ..., w_{k-1}(t), u_{r+1}(t), ..., u_n(t))$, $t \in I''$ is a B-basis of $\mathcal{U}|_{I''}$. ∎

Remark 3.2. Let us observe that the previous theorem shows that, under its hypotheses, the normalized B-basis $(u_0, ..., u_n)$ of $\mathcal{U}$ and the normalized B-basis $(v_0, ..., v_r, w_0, ..., w_s)$ of $\mathcal{V}$ satisfy

$$v_i = u_i, \quad i = 0, ..., r-k; \qquad w_i = u_{n-s+i}, \quad i = k, ..., s. \tag{3.5}$$

Remark 3.3. Let us observe that, if t_0 is a 1-admissible parameter then, the matrix corresponding to the left B-algorithm is $L = I_{r+1}$. Moreover if

$k>1$ and t_0 is a k-admissible parameter in $\mathcal{U}$ then the factorization (2.14) of L is of the form

$$L = L_{k-2}L_{k-3}\cdots L_0$$
$$=\begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \tilde{L}_{k-2} \end{pmatrix}\begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \tilde{L}_{k-3} \end{pmatrix}\cdots\begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \tilde{L}_{0} \end{pmatrix}, \qquad (3.6)$$

where $\tilde{L}=\tilde{L}_{k-2}\tilde{L}_{k-3}\cdots\tilde{L}_0$ is the factorization of type (2.14) of the matrix $\tilde{L}$ of (3.1). In a similar way, the matrix corresponding to the right B-algorithm is $U=I_{s+1}$ for 1-admissible parameters and the factorization (2.15) of U for k-admissible parameters with $k>1$ is of the form

$$U = U_{k-2}U_{k-3}\cdots U_0$$
$$=\begin{pmatrix} \tilde{U}_{k-2} & 0 \\ 0 & I_{s-k+1} \end{pmatrix}\begin{pmatrix} \tilde{U}_{k-3} & 0 \\ 0 & I_{s-k+1} \end{pmatrix}\cdots\begin{pmatrix} \tilde{U}_{0} & 0 \\ 0 & I_{s-k+1} \end{pmatrix}, \qquad (3.7)$$

where $\tilde{U}=\tilde{U}_{k-2}\tilde{U}_{k-3}\cdots\tilde{U}_0$ is the factorization of type (2.15) of the matrix $\tilde{U}$ of (3.1). If $k=1$ then the B-algorithm is symmetric. If $k>1$ and we denote $\{\tilde{u}_l^{(l)}, ..., \tilde{u}_{k-2}^{(l)}, 1, ..., 1\}$ and $\{1, ..., 1, 1-\tilde{l}_l^{(l)}, ..., 1-\tilde{l}_{k-2}^{(l)}\}$ the diagonal entries of $\tilde{U}_l$ and $\tilde{L}_l$ then the B-algorithm will be symmetric if and only if $\tilde{u}_i^{(l)}=\tilde{l}_i^{(l)}$ for all $i=l, ..., k-2$ and $l=0, ..., k-2$.

The following two matricial results will be very useful in Section 5.

LEMMA 3.4. *Let $L=(l_{ij})_{1\leqslant i,j\leqslant k}$ be a $k\times k$ nonsingular stochastic lower triangular matrix. Let $U=(u_{ij})_{1\leqslant i,j\leqslant k}$ be the $k\times k$ upper bidiagonal stochastic matrix such that $u_{ii}=l_{i+1,i}$ for $i=1, ..., k-1$. If $u_{ii}\neq 0$ for all $i=1, ..., k-1$ then U is nonsingular and the matrix $X=(x_{ij})_{1\leqslant i,j\leqslant k}$ defined by $X=LU^{-1}$ satisfies*

$$x_{1k}=(-1)^{k+1}\prod_{j=1}^{k-1}\frac{1-\alpha_j}{\alpha_j}, \qquad x_{ij}=\delta_{i,j+1}, \quad i=2, ..., k, \qquad (3.8)$$

where $\alpha_i := u_{ii}$ for all $i=1, ..., k-1$ and $\delta_{i,j+1}$ is the Kronecker's delta.

Proof. Since L is stochastic and lower bidiagonal $l_{1j}=\delta_{1j}$ and then the first row of X coincides with the first row of U^{-1} and x_{1k} satisfies (3.8). By the definition of X we have that $X[2, ..., k\,|\,1, ..., k]=L[2, ..., k\,|\,1, ..., k]\,U^{-1}=U[1, ..., k-1\,|\,1, ..., k]\,U^{-1}$ and hence we obtain that x_{ij} satisfies (3.8) for all $i=2, ..., k$. ∎

LEMMA 3.5. *Let L (resp., U) be a $k\times k$ nonsingular stochastic totally positive lower (resp., upper) triangular matrix. Let $L=L_{k-2}\cdots L_0$ and $U=U_{k-2}\cdots U_0$ be their factorizations* (2.14) *and* (2.15). *If $u_i^{(l)}=l_i^{(l)}\neq 0$ for* *all* $l=0, ..., k-2$, $i=l, ..., k-2$, *then the* $k\times k$ *matrices* $X_l=(x_{ij}^{(l)})_{1\leqslant i,\, j\leqslant k}$ $(0\leqslant l\leqslant k-2)$ *defined by* $X_l:=L_l\cdots L_0U_0^{-1}\cdots U_l^{-1}$, *satisfy*

$$\begin{aligned}
&\forall i=1, ..., l+1, && x_{ij}^{(l)}=0 \quad \text{if } i+j>k+1,\\
& && x_{i,\,k+1-i}^{(l)}=(-1)^{k+i}\prod_{j=i-1}^{k-2}\frac{1-\alpha_j^{(i-1)}}{\alpha_j^{(i-1)}}\\
&\forall i=l+2, ..., k, && x_{ij}^{(l)}=\delta_{i,\,j+l+1},
\end{aligned}\tag{3.9}$$

where $\alpha_i^{(l)}=u_i^{(l)}$ *for all* $l=0, ..., k-2$, $i=l, ..., k-2$.

Proof. Let us first observe that $L_l=\begin{pmatrix} I_l & 0\\ 0 & \bar{L}_{k-l}\end{pmatrix}$ and $U_l=\begin{pmatrix} \bar{U}_{k-l} & 0\\ 0 & I_l\end{pmatrix}$ where $\bar{L}_{k-l}$ and $\bar{U}_{k-l}$ are $(k-l)\times(k-l)$ matrices satisfying the hypotheses of Lemma 3.4. Since $\alpha_i^l\neq 0$ then U_l is nonsingular. Moreover,

$$U_l^{-1}=\begin{pmatrix} \bar{U}_{k-l}^{-1} & 0\\ 0 & I_l\end{pmatrix}$$

and X_l is well defined. We shall prove (3.9) by induction on l. By Lemma 3.4 X_0 satisfies (3.9). Let us now suppose that the matrix X_{l-1} satisfies (3.9). Let $\bar{X}_l=(\tilde{x}_{ij}^{(l)})_{1\leqslant i,\, j\leqslant k}$ be the matrix defined by $\bar{X}_l:=X_{l-1}U_l^{-1}$. Clearly, $X_l=L_l\bar{X}_l$. By considering the structure of U_l^{-1} and the induction hypothesis we deduce that the last $k-l$ rows of $\bar{X}_l$ are the first $k-l$ rows of U_l^{-1} and that $\bar{X}_l[l+1, ..., k\,|\,1, ..., k-l]=U_{k-l}^{-1}$, $\bar{X}_l[l+1, ..., k\,|\,k-l+1, ..., k]=O_{k-l,\,l}$ where $O_{k-l,\,l}$ is the $(k-l)\times(l)$ null matrix. We also deduce that the last l columns of $\bar{X}_l$ are the last l columns of X_{l-1} and then $\tilde{x}_{ij}^{(l)}=0$ if $i+j>k+1$ and $\tilde{x}_{i,\,k+1-i}^{(l)}=(-1)^{k+i}\prod_{j=i-1}^{k-2}((1-\alpha_j^{(i-1)})/\alpha_j^{(i-1)})$ for all $i=1, ..., l$.

By considering the structure of L_l and $\bar{X}_l$ we deduce that the first l rows of X_l are the first l rows of $\bar{X}_l$ and then $x_{ij}^{(l)}=\tilde{x}_{ij}^{(l)}$ and (3.9) holds for $i=1, ..., l$. Moreover, $X_l[l+1, ..., k\,|\,1, ..., k-l]=L_l[l+1, ..., k\,|\,1, ..., k]$ $\bar{X}_l[1, ..., k\,|\,1, ..., k-l]=\bar{L}_{k-l}\bar{U}_{k-l}^{-1}$ and $X_l[l+1, ..., k\,|\,k-l+1, ..., k]=L_l[l+1, ..., k\,|\,1, ..., k]\,\bar{X}_l[1, ..., k\,|\,k-l+1, ..., k]=O_{k-l,\,k}$. By Lemma 3.4, $\bar{L}_{k-l}\bar{U}_{k-l}^{-1}$ satisfies (3.9) and then $x_{l+1,\,k-l}^{(l)}=(-1)^{k+l+1}\prod_{j=l}^{k-2}((1-\alpha_j^{(l)})/\alpha_j^{(l)})$ and $x_{ij}^{(l)}=\delta_{i,\,j+l+1}$ for $i=l+2, ..., k$. ∎

4. KNOT INSERTION AND B-ALGORITHMS

Let us introduce the basic definitions for knot insertion in a given space $\mathcal{U}$ with shape preserving representations. By $\mathrm{supp}(f)$ we denote the support of a function f.

DEFINITION 4.1. Let $\mathcal{U}$ be a space of functions defined on I with a normalized B-basis and $t_0 \in \mathrm{Int}(I)$. We say that $k := \dim(\mathcal{U}|_{I'}) + \dim(\mathcal{U}|_{I''}) - \dim(\mathcal{U})(\geqslant 0)$ is the *potential knot multiplicity* in $\mathcal{U}$ with t_0.

DEFINITION 4.2. Let $\mathcal{U}^{n+1}$ be an $(n+1)$-dimensional space of functions defined on I with a normalized B-basis and $t_0 \in \mathrm{Int}(I)$ whose potential knot multiplicity in $\mathcal{U}$ is k. Then we say that we perform an *elementary knot insertion* with t_0 if there exits an $(n+2)$-dimensional space $\mathcal{U}^{n+2} \supseteq \mathcal{U}^{n+1}$ with a normalized B-basis such that $\mathcal{U}^{n+2}|_{I'} = \mathcal{U}^{n+1}|_{I'}$ and $\mathcal{U}^{n+2}|_{I''} = \mathcal{U}^{n+1}|_{I''}$.

Remark 4.3. Let us observe that if we perform an elementary knot insertion with t_0 (whose potential knot multiplicity in $\mathcal{U}^{n+1}$ was k) then $k-1$ is the potential knot multiplicity in $\mathcal{U}^{n+2}$ of t_0. So, we can perform at most k consecutive knot insertions with t_0.

Remark 4.4. If $(u_0, ..., u_n)$ is a normalized B-basis of functions continuous at t_0 then, by Remark 2.6, the number of basis functions among $u_0, ..., u_n$ which do not vanish at t_0 coincides with the potential knot multiplicity of t_0 in $\mathcal{U}$.

The following result shows that the space obtained after a maximal number of knot insertions coincides with the space obtained after a B-algorithm.

PROPOSITION 4.5. *Let $\mathcal{U} = \mathcal{U}^{n+1}$ be an $(n+1)$-dimensional space of functions defined on I with a normalized B-basis $(u_0, ..., u_n)$, let $t_0 \in \mathrm{Int}(I)$ whose potential knot multiplicity in $\mathcal{U}$ is k and let $\mathcal{V}$ as in* (2.7). *Let us assume that we can perform k consecutive knot insertions, obtaining the spaces*

$$(\mathcal{U}^{n+1} \subseteq)\ \mathcal{U}^{n+2} \subseteq \cdots \subseteq \mathcal{U}^{n+1+k}. \tag{4.1}$$

Then $\mathcal{U}^{n+1+k} = \mathcal{V}$.

Proof. Let $r+1 = \dim(\mathcal{U}|_{I'})$ and $s+1 = \dim(\mathcal{U}|_{I''})$. By Definitions 4.1 and 4.2, $r+1+s+1 = n+k+1 (= \dim(\mathcal{U}^{n+1+k}))$. Let $(v_0, ..., v_r, w_0, ..., w_s)$ and $(y_0, ..., y_r, z_0, ..., z_s)$ be the normalized B-bases of $\mathcal{V}$ and $\mathcal{U}^{n+1+k}$, respectively. By Definition 4.2, $\mathcal{U}|_{I'} = \mathcal{U}^{n+1+k}|_{I'}$ and $\mathcal{U}|_{I''} = \mathcal{U}^{n+1+k}|_{I''}$. By (2.11), for each $i = 0, ..., r$, $y_i(t) = 0$, $\forall t \in I''$ and for each $j = 0, ..., s$, $z_j(t) = 0$, $\forall t \in I'$ and so $(y_0, ..., y_r)$ and $(z_0, ..., z_s)$ are totally positive bases of $\mathcal{U}^{n+1+k}|_{I'} = \mathcal{U}|_{I'}$ and $\mathcal{U}^{n+1+k}|_{I''} = \mathcal{U}|_{I''}$, respectively. Since $(y_0, ..., y_r, z_0, ..., z_s)$ is a normalized

B-basis $\sum_{i=0}^{r} y_i(t)=1$, $\forall t \in I'$, $\sum_{i=0}^{s} z_i(t)=1$, $\forall t \in I''$ and by Proposition 2.3 it satisfies

$$0=\inf\left\{\frac{y_i(t)}{y_j(t)}\,\middle|\, t\in I,\ y_j(t)\neq 0\right\}=\inf\left\{\frac{y_i(t)}{y_j(t)}\,\middle|\, t\in I',\ y_j(t)\neq 0\right\},\qquad i\neq j,$$

$$0=\inf\left\{\frac{z_k(t)}{z_l(t)}\,\middle|\, t\in I,\ z_l(t)\neq 0\right\}=\inf\left\{\frac{z_k(t)}{z_l(t)}\,\middle|\, t\in I'',\ z_l(t)\neq 0\right\},\qquad k\neq l.$$

Then $(y_0, ..., y_r)$ and $(z_0, ..., z_s)$ satisfy (2.4) and so are the normalized B-basis of $\mathcal{U}|_{I'}$ and $\mathcal{U}|_{I''}$, respectively. By the uniqueness of normalized B-bases (see Proposition 2.5) $v_i=y_i$, $i=0, ..., r$; $w_j=z_j$, $j=0, ..., s$ and hence $\mathcal{U}^{n+1+k}=\mathcal{V}$. ∎

DEFINITION 4.6. Let $\mathcal{U}=\mathcal{U}^{n+1}$ be an $(n+1)$-dimensional space of functions on I with a normalized B-basis and $t_0 \in \operatorname{Int}(I)$ whose potential knot multiplicity in $\mathcal{U}$ is k. Let $(v_0, ..., v_r, w_0, ..., w_s)$ be the normalized B-basis of the space $\mathcal{V}$ defined in (2.7). We say that the B-algorithm corresponding to t_0 (associated to the matrix M of (2.10)) *provides a knot insertion algorithm* if $M=M_{k-1}\cdots M_0$, with M_i an $(n+i+2)\times(n+i+1)$ stochastic bidiagonal totally positive matrix and, for each $p=0, ..., k-1$, the system

$$(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1}) := (v_0, ..., v_r, w_0, ..., w_s)\, M_{k-1}\cdots M_p \qquad (4.2)$$

is the normalized B-basis of its $(n+1+p)$-dimensional generated space $\mathcal{U}^{n+1+p}$.

Remark 4.7. Let us observe that the fact that $\mathcal{V}|_{I'}=\mathcal{U}^{n+1}|_{I'}$ and $\mathcal{V}|_{I''}=\mathcal{U}^{n+1}|_{I''}$ implies that all intermediate spaces $\mathcal{U}^{n+1+p}$ of Definition 4.6 satisfy $\mathcal{U}^{n+1+p}|_{I'}=\mathcal{U}^{n+1}|_{I'}$ and $\mathcal{U}^{n+1+p}|_{I''}=\mathcal{U}^{n+1}|_{I''}$ for all $p=0, ..., k-1$. Thus Definition 4.6 is coherent with Definition 4.2.

Remark 4.8. If the B-algorithm provides a knot insertion algorithm the control polygons $P_0^{n+p+1}\cdots P_{n+p}^{n+p+1}$ (see [16]) of a curve γ with respect to the normalized B-basis of $\mathcal{U}^{n+1+p}$ $(p=0, ..., k-1)$ are related by

$$\begin{aligned}(P_0^{n+p+1}, ..., P_{n+p}^{n+p+1})^T &= M_{p-1}\cdots M_0(P_0^{n+1}, ..., P_n^{n+1})^T\\ &= M_{p-1}(P_0^{n+p}, ..., P_{n+p-1}^{n+p})^T.\end{aligned}$$

Each matrix M_p is bidiagonal, stochastic and totally positive and so it defines a corner cutting algorithm. Let us now suppose that $P_0\cdots P_n$ is the control polygon of a curve γ with respect to the normalized B-basis of $\mathcal{U}$ and that there exist k consecutive corner cutting algorithms providing polygons with an increasing number of points and such that the final polygon is the control polygon of γ with respect to the normalized B-basis of $\mathscr{V}$. Then each algorithm defines a bidiagonal stochastic and totally positive matrix Q_p. The matrix $M := Q_{k-1} \cdots Q_0$ satisfies (2.10) and is so the matrix of the B-algorithm.

The following result shows that elementary knot insertions are always associated to corner cutting algorithms.

THEOREM 4.9. *Let $\mathscr{U} = \mathscr{U}^{n+1}$ be an $(n+1)$-dimensional space of functions on I with a normalized totally positive basis and $t_0 \in \mathrm{Int}(I)$ whose potential knot multiplicity in $\mathscr{U}$ is k. Then the B-algorithm corresponding to t_0 provides a knot insertion algorithm if and only if we can perform k consecutive elementary knot insertions with t_0.*

Proof. If the B-algorithm corresponding to t_0 provides a knot insertion algorithm then we can perform k consecutive elementary knot insertions with t_0 by Definitions 4.2 and 4.6 and Remark 4.7. Let us prove the converse. Let $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ $(p = 0, ..., k-1)$ be the normalized B-bases of the spaces $\mathscr{U}^{n+1+p}$ of (4.1) and let us consider the matrices $M_p = (m_{ij}^{(p)})_{0 \leqslant i \leqslant n+p+1;\, 0 \leqslant j \leqslant n+p}$ satisfying

$$(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1}) = (u_0^{n+p+2}, ..., u_{n+p+1}^{n+p+2})\, M_p. \tag{4.3}$$

Let $I_i^j := \{t \in I \mid u_i^j(t) \neq 0\}$ $(j = n+1, ..., n+k;\ i = 0, ..., j-1)$. By Proposition 2.1, I_i^j is an interval and $\mathrm{supp}(u_i^j) = [\alpha_i^j, \beta_i^j]$ with $\alpha_0^j \leqslant \cdots \leqslant \alpha_{j-1}^j$ and $\beta_0^j \leqslant \cdots \leqslant \beta_{j-1}^j$ for $j = n+1, ..., n+k$. Let $(l_m^{i,j})_{m \in \mathbf{N}}$ (resp., $(r_m^{i,j})_{m \in \mathbf{N}}$) be a monotone decreasing (resp., increasing) sequence such that $l_m^{i,j} \in I_i^j$ and $\lim_{m \to \infty} l_m^{i,j} = \alpha_i^j$ if $\alpha_i^j \notin I_i^j$ or $l_m^{i,j} = \alpha_i^j$, $\forall m \in \mathbf{N}$ if $\alpha_i^j \in I_i^j$ (resp., $r_m^{i,j} \in I_i^j$ and $\lim_{m \to \infty} r_m^{i,j} = \beta_i^j$ if $\beta_i^j \notin I_i^j$ or $r_m^{i,j} = \beta_i^j$, $\forall m \in \mathbf{N}$ if $\beta_i^j \in I_i^j$). By Propositions 2.1(i) and 2.3, if $i_1 < i_2$ we have for any $j = n+1, ..., n+k$

$$\begin{aligned} 0 &= \inf\left\{\frac{u_{i_2}^j(t)}{u_{i_1}^j(t)} \,\middle|\, u_{i_1}^j(t) \neq 0\right\} = \lim_{m \to \infty} \frac{u_{i_2}^j(l_m^{i_1,j})}{u_{i_1}^j(l_m^{i_1,j})}; \\ 0 &= \inf\left\{\frac{u_{i_1}^j(t)}{u_{i_2}^j(t)} \,\middle|\, u_{i_2}^j(t) \neq 0\right\} = \lim_{m \to \infty} \frac{u_{i_1}^j(r_m^{i_2,j})}{u_{i_2}^j(r_m^{i_2,j})}. \end{aligned} \tag{4.4}$$

By Proposition 4.5, $\mathscr{V} = \mathscr{U}^{n+k+1}$ and then by (4.3), it is sufficient to see that each matrix M_p $(0 \leqslant p \leqslant k-1)$ is bidiagonal stochastic and nonnegative. Let $m_{f(i),i}^{(p)}$ and $m_{l(i),i}^{(p)}$ be the first and last nonzero elements of the ith column of M_p. By (4.3),

$$u_i^{n+p+1} = m_{f(i),i}^{(p)} u_{f(i)}^{n+p+2} + \sum_{j=f(i)+1}^{l(i)-1} m_{ji}^{(p)}(p)\, u_j^{n+p+2} + m_{l(i),i}^{(p)} u_{l(i)}^{n+p+2},$$
$$i = 0, ..., n+p. \tag{4.5}$$

Since all functions are nonnegative, we derive from (4.4)

$$0 \leqslant \lim_{m \to \infty} \frac{u_i^{n+p+1}(l_m^{f(i), n+p+2})}{u_{f(i)}^{n+p+2}(l_m^{f(i), n+p+2})} = m_{f(i), i}^{(p)},$$

$$0 \leqslant \lim_{m \to \infty} \frac{u_i^{n+p+1}(r_m^{l(i), n+p+2})}{u_{l(i)}^{n+p+2}(r_m^{l(i), n+p+2})} = m_{l(i), i}^{(p)}$$

and so $m_{f(i), i}^{(p)} > 0$ and $m_{l(i), i}^{(p)} > 0$. Let us take $j > i$. Let us prove that $f(j) > f(i)$. Otherwise, if $f(j) \leqslant f(i)$ then $(m_{f(j), i}^{(p)} / m_{f(j), j}^{(p)}) \geqslant 0$ and there exists $K \in \mathbf{R}$ such that $K > (m_{f(j), i}^{(p)} / m_{f(j), j}^{(p)})$. Let us define the function $g := u_i^{n+p+1} - K u_j^{n+p+1}$. By (4.3), $g = (m_{f(j), i}^{(p)} - K m_{f(j), j}^{(p)})\, u_{f(j)}^{n+p+2} + \sum_{h=f(j)+1}^{n+p+2} (m_{hi}^{(p)} - K m_{hj}^{(p)})\, u_h^{n+p+2}$. From this formula we conclude that $\alpha_{f(j)}^{n+p+2} \leqslant \inf \operatorname{supp}(g)$. We also know that $\alpha_i^{n+p+1} \leqslant \alpha_j^{n+p+1}$ and so $g(t) = 0$, $\forall t \leqslant \alpha_i^{n+p+1}$. From (4.4) we have

$$\lim_{m \to \infty} \frac{g(l_m^{i, n+p+1})}{u_i^{n+p+1}(l_m^{i, n+p+1})} = \lim_{m \to \infty} \frac{u_i^{n+p+1}(l_m^{i, n+p+1}) - K u_j^{n+p+1}(l_m^{i, n+p+1})}{u_i^{n+p+1}(l_m^{i, n+p+1})} = 1.$$

Thus $\alpha_i^{n+p+1} = \inf \operatorname{supp}(g)$ and $g \geqslant 0$ at a neighbourhood of $\inf \operatorname{supp}(g)$. Therefore, from (4.4) and (4.5) we obtain $0 \leqslant \lim_{m \to \infty} (g(l_m^{f(j), n+p+2}) / u_{f(j)}^{n+p+2}(l_m^{f(j), n+p+2})) = m_{f(j), i}^{(p)} - K m_{f(j), j}^{(p)}$, contradicting our choice of K. So $f(j) > f(i)$. With a similar reasoning, we can deduce that $l(i) < l(j)$. Thus the matrix M_p must be a bidiagonal matrix ($m_{ij}^{(p)} = 0$ if $i < j$ or if $i > j+1$) with nonnegative elements and then totally positive.

It remains to see that M_p is stochastic. Let $e = (1, ..., 1)^T \in \mathbf{R}^{n+p+1}$ and $d = (d_0, ..., d_{n+p+1}) := M_p e$. Since $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ is normalized, postmultiplying both sides of (4.3) by e we get $d_0 u_0^{n+p+2} + \cdots + d_{n+p+1} u_{n+p+1}^{n+p+2} = 1$. Since $(u_0^{n+p+2}, ..., u_{n+p+1}^{n+p+2})$ is also normalized, we obtain $d = (1, ..., 1)$ and so M_p is stochastic. ∎

5. EQUIVALENCE BETWEEN SYMMETRIC B-ALGORITHMS AND KNOT INSERTION ALGORITHMS

The following result shows how symmetric B-algorithms are precisely the B-algorithms providing knot insertion.

THEOREM 5.1. *Let $(u_0, ..., u_n)$ be the normalized B-basis of a vector space $\mathcal{U}$ of functions defined on I. Let $t_0 \in \operatorname{Int}(I)$ be a k-admissible parameter in $\mathcal{U}$. Then the B-algorithm is symmetric if and only if we can perform k consecutive knot insertions with t_0 and t_0 is a $(k-p)$-admissible parameter in the corresponding spaces $\mathcal{U}^{n+p+1}$, $p = 1, ..., k-1$.*

Proof. Let us first suppose that the B-algorithm is symmetric. Let $(v_0, ..., v_r)$ and $(w_0, ..., w_s)$ be the systems defined on (2.5) and (2.6), respectively. From Remark 4.4 we know that $k = r + s + 1 - n$ and that k coincides with the potential knot multiplicity with t_0 in $\mathscr{U}$. By Remark 3.3, if $k = 1$ then the matrices L and U corresponding to the left and right B-algorithm are I_{r+1} and I_{s+1} (respectively) moreover for $k > 1$ the factorizations (2.14) and (2.15) of L and U can be written as

$$L = L_{k-2} \cdots L_0 = \begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \tilde{L} \end{pmatrix} = \begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \tilde{L}_{k-2} \end{pmatrix} \cdots \begin{pmatrix} I_{r-k+1} & 0 \\ 0 & \tilde{L}_0 \end{pmatrix},$$

$$U = U_{k-2} \cdots U_0 = \begin{pmatrix} \tilde{U} & 0 \\ 0 & I_{s-k+1} \end{pmatrix} = \begin{pmatrix} \tilde{U}_{k-2} & 0 \\ 0 & I_{s-k+1} \end{pmatrix} \cdots \begin{pmatrix} \tilde{U}_0 & 0 \\ 0 & I_{s-k+1} \end{pmatrix}, \tag{5.1}$$

where $\tilde{L} = \tilde{L}_{k-2} \cdots \tilde{L}_0$ (resp., $\tilde{U} = \tilde{U}_{k-2} \cdots \tilde{U}_0$) is the factorization (2.14) (resp., (2.15)) of the matrix $\tilde{L}$ (resp., $\tilde{U}$) of (3.1). Let $L_{k-1} := I_{r+1}$, $U_{k-1} := I_{s+1}$. Since the B-algorithm is symmetric the diagonal entries $\{\tilde{u}_p^{(p)}, ..., \tilde{u}_{k-2}^{(p)}, 1, ..., 1\}$, $\{1, ..., 1, 1 - \tilde{l}_p^{(p)}, ..., 1 - \tilde{l}_{k-2}^{(p)}\}$ of $\tilde{U}_p$ and $\tilde{L}_p$ (respectively) satisfy $\tilde{u}_i^{(p)} = \tilde{l}_i^{(p)}$ for each $i = p, ..., k-2$ and we can define $M_p = (m_{ij}^{(p)})_{0 \leqslant i \leqslant n+p+1;\, 0 \leqslant j \leqslant n+p}$ $(0 \leqslant p \leqslant k-1)$ as the $(n+p+2) \times (n+p+1)$ bidiagonal matrix such that

$$M_p[1, ..., r+1] := L_p,$$

$$M_p[n+p+2-s, ..., n+p+2 \,|\, n+p+1-s, ..., n+p+1] := U_p.$$

Since L_p and U_p $(p = 0, ..., k-1)$ are nonsingular, bidiagonal, stochastic totally positive matrices then M_p $(p = 0, ..., k-1)$ is also stochastic, totally positive and its column rank is $n+p+1$. Let $(v_0, ..., v_r, w_0, ..., w_s)$ be the normalized B-basis of the space $\mathscr{V}$ defined in (2.7). Let $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ $(p = 0, ..., k-1)$ be the system defined by

$$(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1}) := (v_0, ..., v_r, w_0, ..., w_s)\, \bar{M}_p, \qquad \bar{M}_p := M_{k-1} \cdots M_p, \tag{5.2}$$

and let $\mathscr{U}^{n+p+1}$ be its generated space of functions on I. Clearly,

$$\mathscr{V} \supset \mathscr{U}^{n+k} \supset \mathscr{U}^{n+k-1} \supset \cdots \supset \mathscr{U}^{n+2} \supset \mathscr{U}^{n+1} = \mathscr{U}. \tag{5.3}$$

Since by property (A) of Definition 2.9, $\mathscr{U}$ is $\mathscr{C}^0$ in $(t_0 - \varepsilon, t_0 + \varepsilon)$ then, $\mathscr{U}|_{I'}$ is $\mathscr{C}^0$ in $I' \cap (t_0 - \varepsilon, t_0 + \varepsilon)$, (resp., $\mathscr{U}|_{I''}$ is $\mathscr{C}^0$ in $I'' \cap (t_0 - \varepsilon, t_0 + \varepsilon)$) and the functions $v_0, ..., v_r$ (resp., $w_0, ..., w_s$) are continuous on $(t_0 - \varepsilon, t_0]$ (resp.,

$(t_0, t_0+\varepsilon)$). Taking into account that $(v_0, ..., v_r)$ and $(w_0, ..., w_s)$ are normalized and, using Corollary 2.12, (2.5) and (2.6), we derive

$$v_i(t_0)=0, \qquad i=0, ..., r-1, \qquad v_r(t_0)=1,$$

$$\lim_{t\to t_0+} w_0(t)=1, \qquad \lim_{t\to t_0+} w_i(t)=0, \quad i=1, ..., s.$$

By (5.2), $(u_0^{n+k}, ..., u_{n+k-1}^{n+k})=(v_0, ..., v_r, w_0, ..., w_s)\, M_{k-1}=(v_0, ..., v_r+w_0, ..., w_s)$ and then one can easily deduce that $\mathcal{U}^{n+k}$ is $\mathcal{C}^0$ in $(t_0-\varepsilon, t_0+\varepsilon)$, i.e.,

$$\mathcal{U}^{n+k}\subseteq\mathcal{C}(t_0-\varepsilon, t_0+\varepsilon). \tag{5.4}$$

From (5.3) we know that $\mathcal{U}^{n+p+1}$ $(p=0, ..., k-1)$ is formed by continuous functions on $(t_0-\varepsilon, t_0+\varepsilon)$.

Let us now prove that the system $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ is the normalized B-basis of $\mathcal{U}^{n+p+1}$ for all $p=0, ..., k-1$.

The system $(v_0, ..., v_r, w_0, ..., w_s)$ is the normalized B-basis of $\mathcal{V}$ and the matrix $\bar{M}_p$ of (5.2) is stochastic and totally positive because it is the product of stochastic and totally positive matrices. Then from (5.2) we deduce that $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ is a normalized and totally positive basis of $\mathcal{U}^{n+p+1}$. Since the column rank of $\bar{M}_p$ is $n+p+1$ then the dimension of $\mathcal{U}^{n+p+1}$ is $n+p+1$.

Let $I_j^i:=[\alpha_j^i, \beta_j^i]=\operatorname{supp}(u_j^i)$. Let $(l_m^{j,i})_{m\in\mathbf{N}}$ (resp., $(r_m^{j,i})_{m\in\mathbf{N}}$) be a monotone decreasing (resp., increasing) sequence such that $l_m^{j,i}\in I_j^i$ and $\lim_{m\to\infty} l_m^{j,i}=\alpha_j^i$ if $\alpha_j^i\notin I_j^i$ or $l_m^{j,i}=\alpha_j^i$, $\forall m\in\mathbf{N}$ if $\alpha_j^i\in I_j^i$ (resp., $r_m^{j,i}\in I_j^i$ and $\lim_{m\to\infty} r_m^{j,i}=\beta_j^i$ if $\beta_j^i\notin I_j^i$ or $r_m^{j,i}=\beta_j^i$, $\forall m\in\mathbf{N}$ if $\beta_j^i\in I_j^i$). From Propositions 2.1(i) and 2.3 we derive

$$\inf\left\{\frac{u_i^{n+p+1}(t)}{u_j^{n+p+1}(t)}\,\middle|\, u_j^{n+p+1}(t)\neq 0\right\}=\lim_{m\to\infty}\frac{u_i^{n+p+1}(r_m^{j,n+p+1})}{u_j^{n+p+1}(r_m^{j,n+p+1})},$$

$$0\leqslant i<j\leqslant n+p;$$

$$\inf\left\{\frac{u_i^{n+p+1}(t)}{u_j^{n+p+1}(t)}\,\middle|\, u_j^{n+p+1}(t)\neq 0\right\}=\lim_{m\to\infty}\frac{u_i^{n+p+1}(l_m^{j,n+p+1})}{u_j^{n+p+1}(l_m^{j,n+p+1})},$$

$$0\leqslant j<i\leqslant n+p.$$

We are going to see now that if $(u_0^{n+p+2}, ..., u_{n+p+1}^{n+p+2})$ is a B-basis (and by Proposition 2.3 satisfies (2.4)) then $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ is also a B-basis. By construction $u_i^{n+p+1}=m_{ii}^{(p)}u_i^{n+p+2}+m_{i+1,i}^{(p)}u_{i+1}^{n+p+2}$ and $m_{i+1,i}^{(p)}=0$ for each $i=0, ..., n+p-s-1$, $m_{ii}^{(p)}=0$ for each $i=r+1, ..., n+p$ and $0<m_{ii}^{(p)}<1$ for each $i=n+p-s, ..., r$. If we denote by $q:=n+p+1$ then $(u_0^{q+1}, ..., u_q^{q+1})$ satisfies (2.4) and we can write

$$\lim_{m\to\infty}\frac{u_i^q(r_m^{j,q})}{u_j^q(r_m^{j,q})}=\begin{cases}\lim\limits_{m\to\infty}\dfrac{m_{ii}^{(p)}\dfrac{u_i^{i+1}(r_m^{j+1,q+1})}{u_{j+1}^{q+1}(r_m^{j+1,q+1})}+m_{i+1,i}^{(p)}\dfrac{u_{i+1}^{q+1}(r_m^{j+1,q+1})}{u_{j+1}^{q+1}(r_m^{j+1,q+1})}}{m_{jj}^{(p)}\dfrac{u_j^{q+1}(r_m^{j+1,q+1})}{u_{j+1}^{q+1}(r_m^{j+1,q+1})}+m_{j+1,j}^{(p)}}=0, & \\ \quad\text{if}\quad m_{j+1,j}^{(p)}\neq 0, & \\ \lim\limits_{m\to\infty}\dfrac{u_i^{q+1}(r_m^{j,q+1})}{u_j^{q+1}(r_m^{j,q+1})}=0, & \\ \quad\text{if}\quad m_{j+1,j}^{(p)}=0, & \end{cases}$$

for all $0\leqslant i<j\leqslant q-1$,

$$\lim_{m\to\infty}\frac{u_i^q(t)}{u_j^q(t)}=\begin{cases}\lim\limits_{m\to\infty}\dfrac{u_{i+1}^{n+p+2}(l_m^{j+1,q+1})}{u_{j+1}^{n+p+2}(l_m^{j+1,q+1})}=0, & \\ \quad\text{if}\quad m_{jj}^{(p)}=0, & \\ \lim\limits_{m\to\infty}\dfrac{m_{ii}^{(p)}\dfrac{u_i^{q+1}(l_m^{j,q+1})}{u_j^{q+1}(l_m^{j,q+1})}+m_{i+1,i}^{(p)}\dfrac{u_{i+1}^{q+1}(l_m^{j,q+1})}{u_j^{q+1}(l_m^{j,q+1})}}{m_{jj}^{(p)}+m_{j+1,j}^{(p)}\dfrac{u_j^{q+1}(l_m^{j,q+1})}{u_j^{q+1}(l_m^{j,q+1})}}=0, & \\ \quad\text{if}\quad m_{jj}^{(p)}\neq 0, & \end{cases}$$

for all $0\leqslant j<i\leqslant q-1$. Then $(u_0^q, ..., u_{q-1}^q)$ also satisfies (2.4) and therefore by Proposition 2.3 is a B-basis of $\mathscr{U}^q$. We can now conclude that each system $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ $p=0, ..., k-1$ is a B-basis of $\mathscr{U}^{n+p+1}$.

It can be checked that the matrix $\bar{M}_p=(\bar{m}_{ij}^{(p)})_{0\leqslant i\leqslant r+s+1;\,0\leqslant j\leqslant n+p}$ of (5.2) satisfies

$$\begin{aligned}&\bar{m}_{ij}^{(p)}=0, && \text{if}\quad i<j,\quad 0\leqslant i\leqslant r\\ &&&\text{or if}\quad i>j+k-p,\quad r+1\leqslant i\leqslant r+s+1;\\ &\bar{m}_{ij}^{(p)}=0, && \text{if}\quad i>j,\quad 0\leqslant j\leqslant r-k+p\\ &&&\text{or if}\quad i<j+k-p,\quad r+1\leqslant j\leqslant n+p;\\ &&&\bar{M}_p[1, ..., r+1]=L_{k-1}\cdots L_p;\\ &\bar{M}_p[r+2, ..., r+2+s\,|\,n+p+1-s, ..., n+p+1]=U_{k-1}\cdots U_p.\end{aligned}\tag{5.5}$$

In particular $\bar{M}_0[1, ..., r+1] = L$, $\bar{M}_0[r+2, ..., r+2+s \mid n+1-s, ..., n+1] = U$, hence $\bar{M}_0$ is the matrix of the B-algorithm and by (2.10), $\mathscr{U}^{n+1} = \mathscr{U}$. We can write now from (5.3)

$$\mathscr{V} \supset \mathscr{U}^{n+k} \supset \mathscr{U}^{n+k-1} \supset \cdots \supset \mathscr{U}^{n+2} \supset \mathscr{U}. \tag{5.6}$$

Since by (2.5) and (2.6), for each $i = 0, ..., r$, $v_i(t) = 0$, $\forall t \in I''$ and for each $i = 0, ..., s$, $w_i(t) = 0$, $\forall t \in I'$ and taking into account (5.2) and (5.5) we can deduce that

$$\begin{gathered}
(u_0^{n+p+1}(t), ..., u_r^{n+p+1}(t)) = (v_0(t), ..., v_r(t))\, L_{k-1} \cdots L_p, \qquad \forall t \in I';\\
u_i^{n+p+1}(t) = 0, \qquad \forall t \in I', \qquad i = r+1, ..., n+p,\\
(u_{n+p-s}^{n+p+1}(t), ..., u_{n+p}^{n+p+1}(t)) = (w_0(t), ..., w_s(t))\, U_{k-1} \cdots U_p, \qquad \forall t \in I'';\\
u_i^{n+p+1}(t) = 0, \qquad \forall t \in I'', \qquad i = 0, ..., n+p-s-1.
\end{gathered} \tag{5.7}$$

Therefore

$$\mathscr{V}|_{I'} = \mathscr{U}^{n+p+1}|_{I'} = \mathscr{U}|_{I'}, \qquad \mathscr{V}|_{I''} = \mathscr{U}^{n+p+1}|_{I''} = \mathscr{U}|_{I''}, \qquad p = 1, ..., k-1, \tag{5.8}$$

and we can perform k consecutive knot insertions with t_0. Since $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ is a normalized B-basis of functions continuous at t_0 satisfying (5.7) we can deduce as in Remark 2.6 that the number of basis functions among $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ which do not vanish at t_0 is $k-p$.

Let us now prove that for each $p = 1, ..., k-1$, t_0 is a $(k-p)$-admissible parameter in $\mathscr{U}^{n+p+1}$. By property (A) of Definition 2.9, we know that there exists $\varepsilon > 0$ such that $\mathscr{U} \subseteq \mathscr{C}^{k-1}(t_0 - \varepsilon, t_0 + \varepsilon)$. We are going to see first that $\mathscr{U}^{n+p+1} \subseteq \mathscr{C}^{k-p-1}(t_0 - \varepsilon, t_0 + \varepsilon)$ for $p = 1, ..., k-1$.

The functions in $\mathscr{U}|_{I'}$ (resp., $\mathscr{U}|_{I''}$) have continuous derivatives up to order $k-1$ in $(t_0 - \varepsilon, t_0 + \varepsilon) \cap I'$ (resp., $(t_0 - \varepsilon, t_0 + \varepsilon) \cap I''$). Then from (5.8)

$$\mathscr{U}^{n+p+1} \subseteq \mathscr{C}^{k-1}(t_0 - \varepsilon, t_0) \cap \mathscr{C}^{k-1}(t_0, t_0 + \varepsilon), \qquad p = 1, ..., k-1. \tag{5.9}$$

Let us suppose that $\mathscr{U}^{n+p+2} \subseteq \mathscr{C}^{k-p-2}(t_0 - \varepsilon, t_0 + \varepsilon)$ and let us prove that the derivatives of the functions in $\mathscr{U}^{n+p+1}$ are continuous up to order $i := k-p-1$ $(0 \leqslant i \leqslant k-2)$. Taking into account that $\mathscr{U}^{n+p+1}$ is a subspace of $\mathscr{U}^{n+p+2}$ we conclude that $\mathscr{U}^{n+p+1} \subseteq \mathscr{C}^{i-1}(t_0 - \varepsilon, t_0 + \varepsilon)$. By (5.9), $\mathscr{U}^{n+p+1} \subseteq \mathscr{C}^{i}(t_0 - \varepsilon, t_0) \cap \mathscr{C}^{i}(t_0, t_0 + \varepsilon)$ and then we only have to see that

$$\lim_{t \to t_0+} u_j^{n+p+1\,(i)}(t) = u_j^{n+p+1\,(i)}(t_0) = \lim_{t \to t_0-} u_j^{n+p+1\,(i)}(t), \qquad j = 0, ..., n+p. \tag{5.10}$$

From (5.2) we can write

$$\lim_{t \to t_0} (u_0^{n+p+1\,(i)}(t), ..., u_{n+p}^{n+p+1\,(i)}(t))$$
$$= \lim_{t \to t_0} (v_0^{(i)}(t), ..., v_r^{(i)}(t), w_0^{(i)}(t), ..., w_s^{(i)}(t))\, \bar{M}_p. \tag{5.11}$$

By (2.6), $\lim_{t \to t_0-} w_j^{(i)}(t) = 0$ for all $j = 0, ..., s$, and from (2.5) and Corollary 2.12 we get $\lim_{t \to t_0-} v_j^{(i)}(t) = v_j^{(i)}(t_0) = 0$ for all $j = 0, ..., r-i-1$. From (5.5) and (5.11) we can deduce now that

$$\begin{aligned}
&\lim_{t \to t_0-} u_j^{n+p+1\,(i)}(t) \\
&\quad = u_j^{n+p+1\,(i)}(t_0) = 0, \qquad j = 0, ..., r-i-1, \quad j = r+1, ..., n+p, \\
&\lim_{t \to t_0-} (u_{r-i}^{n+p+1\,(i)}(t), ..., u_r^{n+p+1\,(i)}(t)) \\
&\quad = (v_{r-i}^{(i)}(t_0), ..., v_r^{(i)}(t_0))\, \bar{M}_p[r-i+1, ..., r+1],
\end{aligned} \tag{5.12}$$

where $\bar{M}_p[r-i+1, ..., r+1] = \bar{L}_p[k-i, ..., k]$ and $\bar{L}_p$ is the $k \times k$ matrix defined by $\bar{L}_p := I_{k-1} \tilde{L}_{k-2} \cdots \tilde{L}_p$.

Analogously, by (2.5), $\lim_{t \to t_0+} v_j^{(i)}(t) = 0$ for all $j = 0, ..., r$ and from (2.6) and Corollary 2.12 we obtain $\lim_{t \to t_0+} w_j^{(i)}(t) = 0$ for all $j = i+1, ..., s$. From (5.5) and (5.11) we can deduce now that

$$\begin{aligned}
&\lim_{t \to t_0+} u_j^{n+p+1\,(i)}(t) = 0, \qquad j = 0, ..., r-i-1, \quad j = r+1, ..., n+p, \\
&\lim_{t \to t_0+} (u_{r-i}^{n+p+1\,(i)}(t), ..., u_r^{n+p+1\,(i)}(t)) \\
&\quad = \lim_{t \to t_0+} (w_0^{(i)}(t), ..., w_i^{(i)}(t))\, \bar{M}_p[r+2, ..., r+i+2 \,|\, r-i+1, ..., r+1],
\end{aligned} \tag{5.13}$$

and $\bar{M}_p[r+2, ..., r+i+2 \,|\, r-i+1, ..., r+1] = \bar{U}_p[1, ..., i+1]$ and $\bar{U}_p$ is the $k \times k$ matrix defined by $\bar{U}_p := I_{k-1} \tilde{U}_{k-2} \cdots \tilde{U}_p$. Taking into account that $\mathscr{U} \subseteq \mathscr{C}^{k-1}(t_0 - \varepsilon, t_0 + \varepsilon)$ and Theorem 3.1 we deduce that $\lim_{t \to t_0+} (w_0^{(i)}(t), ..., w_{k-1}^{(i)}(t)) = (u_{n-s}^{(i)}(t_0), ..., u_r^{(i)}(t_0))\, \tilde{U}^{-1}$ and that $(u_{n-s}^{(i)}(t_0), ..., u_r^{(i)}(t_0)) = (v_{n-s}^{(i)}(t_0), ..., v_r^{(i)}(t_0))\, \tilde{L}$. Therefore

$$\begin{aligned}
\lim_{t \to t_0+} (w_0^{(i)}(t), ..., w_{k-1}^{(i)}(t))\, \bar{U}_p &= (v_{n-s}^{(i)}(t_0), ..., v_r^{(i)}(t_0))\, \tilde{L}\tilde{U}^{-1}\bar{U}_p \\
&= (0, ..., 0, v_{r-i}^{(i)}(t_0), ..., v_r^{(i)}(t_0))\, \tilde{L}\tilde{U}^{-1}\bar{U}_p.
\end{aligned} \tag{5.14}$$

If we define $\tilde{M} := \tilde{L}_{p-1} \cdots \tilde{L}_0 \tilde{U}_0^{-1} \cdots \tilde{U}_{p-1}^{-1}$, then $\tilde{L}\tilde{U}^{-1}\bar{U}_p = \bar{L}_p \tilde{M}$. Since the B-algorithm is symmetric we can apply Lemma 3.5 and then obtain that $\tilde{M}[k-i, ..., k \mid 1, ..., 1+i] = I_{i+1}$ and $\tilde{M}[k-i, ..., k \mid i+2, ..., k]$ is a null matrix. Then from (5.14) we have that

$$\lim_{t \to t_0+} (w_0^{(i)}(t), ..., w_i^{(i)}(t))\, \bar{U}_p[1, ..., i+1]$$

$$= (v_{r-i}^{(i)}(t_0), ..., v_r^{(i)}(t_0))\, \bar{L}_p[k-i, ..., k]. \tag{5.15}$$

From (5.12), (5.13), and (5.15) we conclude that (5.10) holds and so $\mathscr{U}^{n+p+1} \subseteq \mathscr{C}^{k-p-1}(t_0-\varepsilon, t_0+\varepsilon)$. By (5.4), $\mathscr{U}^{n+k} \subseteq \mathscr{C}(t_0-\varepsilon, t_0+\varepsilon)$, and we can now conclude that $\mathscr{U}^{n+k-1} \subseteq \mathscr{C}^1(t_0-\varepsilon, t_0+\varepsilon)$. Iterating the previous reasoning we obtain that $\mathscr{U}^{n+p+1} \subseteq \mathscr{C}^{k-p-1}(t_0-\varepsilon, t_0+\varepsilon)$ for $p=1, ..., k-1$.

Since t_0 is a k-admissible parameter in $\mathscr{U}$ and hence satisfies property (B) of Definition 2.9, by Remark 2.10 and (5.6), we can deduce that there exist $k-p$ functions $f_i \in \mathscr{U}^{n+p+1}$ $(i=0, ..., k-p-1)$ satisfying $f_i^{(j)}(t_0)=0$, $j=0, ..., i-1$; $f_i^{(i)}(t_0) \neq 0$ and deduce from Remark 2.10 that property (B) holds in $\mathscr{U}^{n+p+1}$. Therefore t_0 is a $(k-p)$-admissible parameter.

Let us now see the converse. Let us assume that there exist k spaces of functions $\mathscr{U}^{n+k+1} \supset \mathscr{U}^{n+k} \supset \mathscr{U}^{n+k-1} \supset \cdots \supset \mathscr{U}^{n+2} \supset \mathscr{U}^{n+1} (:= \mathscr{U})$ with a normalized B-basis such that t_0 is a $(k-p)$-admissible parameter of $\mathscr{U}^{n+p+1}$ $(p=1, ..., k-1)$ and

$$\mathscr{U}^{n+p+1}|_{I'} = \mathscr{U}^{n+1}|_{I'}, \qquad \mathscr{U}^{n+p+1}|_{I''} = \mathscr{U}^{n+1}|_{I''}, \qquad p=1, ..., k,$$

and let us prove that the B-algorithm is symmetric. By Proposition 4.5, $\mathscr{U}^{n+k+1} = \mathscr{V}$ and therefore

$$u_i^{n+k+1} = v_i, \quad i=0, ..., r; \qquad u_i^{n+k+1} = w_{i-r-1}; \qquad i=r+1, ..., n+k. \tag{5.16}$$

Let $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1})$ be the normalized B-basis of $\mathscr{U}^{n+p+1}$. By Theorem 4.9, the matrix $M_p = (m_{ij}^{(p)})_{0 \leqslant i \leqslant n+p+1;\, 0 \leqslant j \leqslant n+p}$ of (4.2) is bidiagonal stochastic and totally positive. Let us observe that $(u_0^{n+p+1}, ..., u_{n+p}^{n+p+1}) = (u_0^{n+p+2}, ..., u_{n+p+1}^{n+p+2})\, M_p$ $(p=0, ..., k-1)$. Since t_0 is a $(k-p)$-admissible parameter of $\mathscr{U}^{n+p+1}$ for each $p=1, ..., k-1$, we deduce from Remark 3.2 that

$$\begin{aligned} u_i^{n+p+1} &= v_i, && i=0, ..., r-k+p; \\ u_i^{n+p+1} &= w_{i-n-p+s}, && i=r+1, ..., n+p. \end{aligned} \tag{5.17}$$

From (5.16) and (5.17) we have, for each $p=0, ..., k-1$, that $u_i^{n+p+1} = u_i^{n+p+2}$, $i=0, ..., r-k+p$, and $u_i^{n+p+1} = u_{i+1}^{n+p+2}$, $i=r+1, ..., n+p$. Therefore

$$m_{ii}^{(p)} = 1, \quad i=0, ..., r-k+p, \qquad m_{i+1,i}^{(p)} = 1, \quad i=r+1, ..., n+p. \tag{5.18}$$

for each $p=0, ..., k-1$. In particular,

$$M_{k-1}[1, ..., r+1]=I_{r+1},$$
$$M_{k-1}[r+2, ..., n+k+1 \mid r+1, ..., n+k]=I_{s+1}. \tag{5.19}$$

Taking into account that, by (2.5) and (2.6), $v_i(t)=0$, $\forall t \in I''$ $(i=0, ..., r)$ and $w_i(t)=0$, $\forall t \in I'$ $(i=0, ..., s)$, that $(u_0, ..., u_n)=(v_0, ..., v_r, w_0, ..., w_s)$ $M_{k-1} \cdots M_0$ and (5.19), we deduce that $(u_0, ..., u_r)=(v_0, ..., v_r)$ $M_{k-2}[1, ..., r+1] \cdots M_0[1, ..., r+1]$, $\forall t \in I'$ and $L=M_{k-2}[1, ..., r+1] \cdots M_0[1, ..., r+1]$ is the factorization (2.14) of the matrix of the left B-algorithm. Analogously, $(u_{n-s}, ..., u_n)=(w_0, ..., w_s)$ $M_{k-2}[n+k-s, ..., n+k \mid n+k-s-1, ..., n+k-1] \cdots M_0[n+2-s, ..., n+2 \mid n+1-s, ..., n+1]$ and $U=M_{k-2}[n+k-s, ..., n+k \mid n+k-s-1, ..., n+k-1] \cdots M_0[n+2-s, ..., n+2 \mid n+1-s, ..., n+1]$ is the factorization (2.15) of the matrix of the right B-algorithm and hence the B-algorithm is symmetric. ∎

6. B-ALGORITHMS CORRESPONDING TO ADMISSIBLE PARAMETERS ARE SYMMETRIC

In the following theorem we shall see that B-algorithms corresponding to admissible parameters are always symmetric and so, by the results of previous sections, always provide a knot insertion algorithm.

Theorem 6.1. *Let $\mathcal{U}$ be a vector space of functions defined on I with a normalized totally positive basis. If $t_0 \in \mathrm{Int}(I)$ is a k-admissible parameter in $\mathcal{U}$ then its corresponding B-algorithm is symmetric.*

Proof. By Proposition 2.5, $\mathcal{U}$ has a unique normalized B-basis $(u_0, ..., u_n)$. By Remark 3.3, if $k=1$ the B-algorithm is symmetric. Let us now suppose that $k>1$ and let $L=L_{k-2} \cdots L_0$ be the factorization (3.6) of the matrix L of (2.12) associated to the left B-algorithm. Let us first assume that the diagonal entries of L_l $(l=0, ..., k-2)$, given by $\{1, ..., 1, 1-l_l^{(l)}, ..., 1-l_{k-2}^{(l)}\}$, satisfy $l_i^{(l)} \neq 0$ for all $i=l, ..., k-2$. Let us define $\tilde{V}_l$ as the $k \times k$ upper bidiagonal stochastic matrix with diagonal entries $\{l_l^{(l)}, ..., l_{k-2}^{(l)}, 1, ..., 1\}$. Let $\tilde{V}:=\tilde{V}_{k-2} \cdots \tilde{V}_0$. Since $l_i^{(l)} \neq 0$ for all $i=l, ..., k-2$ then $\tilde{V}_l$ $(l=0, ..., k-2)$ is nonsingular and we can define $\tilde{T}=(\tilde{t}_{ij})_{1 \leqslant i, j \leqslant k}$ as $\tilde{T}:=\tilde{L}_{k-2} \cdots \tilde{L}_0 \tilde{V}_0^{-1} \cdots \tilde{V}_{k-2}^{-1}$. It can be easily checked that $\tilde{T}$ satisfies the hypotheses of Lemma 3.5 and then it satisfies

$$\tilde{t}_{ij}=0, \quad i+j>k+1; \qquad \tilde{t}_{i, k+1-i}=(-1)^{k+i} \prod_{j=i-1}^{k-2} \frac{1-l_j^{(i-1)}}{l_j^{(i-1)}}. \tag{6.1}$$

Let V be the $(s+1)\times(s+1)$ matrix given by $V=\begin{pmatrix}\tilde V & 0\\ 0 & I_{s-k+1}\end{pmatrix}$. By construction $\tilde V$ is stochastic and so V is also stochastic. Let us define the system $(z_0, ..., z_s)$ as

$$(z_0, ..., z_s) := (u_{n-s}, ..., u_n)\, V^{-1}. \tag{6.2}$$

If we see that $(z_0, ..., z_s)$ is the normalized B-basis of $\mathscr{U}|_{I''}$ then V is the matrix associated to the right B-algorithm. In this case the B-algorithm is symmetric and the result follows. Since by (2.12), $u_0(t) = \cdots = u_{n-s-1}(t) = 0$, $\forall t \in I''$, we deduce from $1 = \sum_{i=0}^{n} u_i(t)$, $\forall t \in I$ that $(u_{n-s}, ..., u_n)$ is normalized on I''. Let $e = (1, ..., 1)^T \in \mathbf{R}^{s+1}$. Using that V is stochastic we derive from (6.2) that $z_0(t) + \cdots + z_s(t) = (z_0(t), ..., z_s(t))\, Ve = (u_{n-s}(t), ..., u_n(t))\, e = u_{n-s}(t) + \cdots + u_n(t) = 1$, $\forall t \in I''$, and therefore $(z_0, ..., z_s)$ is normalized on I''. Thus it remains to see that $(z_0, ..., z_s)$ is a B-basis of $\mathscr{U}|_{I''}$. Since $(u_0, ..., u_n)$ is a B-basis, the functions $u_i \in R^{n-i}(\mathscr{U})$ $\forall i$ and, by (2.9), $u_i \in R^{n-i}(\mathscr{U}|_{I''})$. Each function z_i is a linear combination of $u_{n-s}, ..., u_i$ because V^{-1} is upper triangular. By (2.3),

$$z_i \in R^{n-i}(\mathscr{U}|_{I''}) \subseteq R^{s-i}(\mathscr{U}|_{I''}), \qquad i = 0, ..., s. \tag{6.3}$$

Since $V^{-1} = \begin{pmatrix}\tilde V^{-1} & 0\\ 0 & I_{s-k+1}\end{pmatrix}$ and using that by Remark 2.6, $n-s+k-1 = r$ we have that

$$z_k(t) = u_{n-s+k}(t) = u_{r+1}(t), ..., z_s(t) = u_n(t), \qquad t \in I'' \tag{6.4}$$

and that

$$(z_0(t), ..., z_{k-1}(t)) = (u_{n-s}(t), ..., u_r(t))\, \tilde V^{-1}, \qquad t \in I''. \tag{6.5}$$

Let $(v_0, ..., v_r, w_0, ..., w_s)$ be the normalized B-basis of $\mathscr{V}$. In particular $(w_0(t), ..., w_s(t))$, $t \in I''$ is the normalized B-basis of $\mathscr{U}|_{I''}$. By (6.4) and Remark 3.2, for each $j = k, ..., s$, $z_j(t) = w_j(t)$, $\forall t \in I''$. Therefore we have that

$$z_j \in L^j(\mathscr{U}|_{I''}), \qquad j = k, ..., s. \tag{6.6}$$

By property (A) of Definition 2.9, the functions in $\mathscr{U}$ are $\mathscr{C}^{(k-1)}$ in $(t_0-\varepsilon, t_0+\varepsilon)$ and then, given $j \in \{0, ..., k-1\}$, we deduce from (6.5) and (3.2) that

$$\begin{aligned}(z_0^{(j)}(t_0), ..., z_{k-1}^{(j)}(t_0)) &= \lim_{t\to t_0^+} ((u_{n-s}^{(j)}(t), ..., u_r^{(j)}(t))\, \tilde V^{-1})\\ &= \lim_{t\to t_0^-} ((v_{r-k}^{(j)}(t), ..., v_r^{(j)}(t))\, \tilde L \tilde V^{-1})\\ &= (v_{r-k}^{(j)}(t_0), ..., v_r^{(j)}(t_0))\, \tilde T. \end{aligned}\tag{6.7}$$

We know from Corollary 2.12 that

$$v_{r-l}^{(j)}(t_0)=0, \qquad \forall l>j, \qquad v_{r-j}^{(j)}(t_0)\neq 0. \tag{6.8}$$

Taking into account (6.8) and (6.1), we deduce from (6.7) that

$$z_i^{(j)}(t_0)=0, \qquad \forall i>j \tag{6.9}$$

and $z_j^{(j)}(t_0)=v_{r-j}^{(j)}(t_0)\,\tilde{t}_{k-j,\,j+1}$. Since the functions v_i's are nonnegative, from (6.8) and Taylor formula we deduce that $(-1)^{(j)}\,v_{r-j}^{(j)}(t_0)>0$. By (6.1), the sign of $\tilde{t}_{k-j,\,j+1}$ is strict and coincides with $(-1)^j$. Therefore

$$z_j^{(j)}(t_0)>0. \tag{6.10}$$

Since (6.9) holds for all $j\in\{0,\ldots,k-1\}$, we deduce from Proposition 2.11 that

$$z_j\in L^j(\mathcal{U}\,|_{I''}), \qquad j=0,\ldots,k-1. \tag{6.11}$$

Therefore we can summarize (6.3), (6.6), and (6.11) with

$$z_i\in R^{s-i}(\mathcal{U}\,|_{I''})\cap L^i(\mathcal{U}\,|_{I''}), \qquad i=0,\ldots,s. \tag{6.12}$$

By Definition 2.2, $w_i\in R^{s-i}(\mathcal{U}\,|_{I''})\cap L^i(\mathcal{U}|_{I''})$, $i=0,\ldots,s$. By Corollary 3.9(i) of [5], $\dim(R^{s-i}(\mathcal{U}|_{I''})\cap L^i(\mathcal{U}\,|_{I''}))=1$ for all $i=0,\ldots,s$ and from (6.12) we deduce that each $z_i=\alpha w_i$ and then z_i is either nonnegative or nonpositive. By (6.4), the functions $z_k,\ldots,z_s$ are nonnegative on I''. From (6.10) and Taylor formula we deduce that the functions z_i $(i=0,\ldots,k-1)$ are nonnegative on a neighbourhood to the right of t_0 and therefore they are nonnegative on I''. By Corollary 3.9(ii) of [5], $(z_0,\ldots,z_s)$ is a B-basis of $\mathcal{U}|_{I''}$.

Let us now suppose that there exists $l_{i_0}^{(l_0)}=0$ for some $l_0\in\{0,\ldots,k-2\}$, $i_0\in\{l_0,\ldots,k-2\}$ and we shall obtain a contradiction. Let $\tau_m:=1/m$, $m\in\mathbf{N}$. Let us define $\tilde{L}_l(\tau_m)$ $(l=0,\ldots,k-2)$ as the $k\times k$ lower bidiagonal stochastic matrix whose diagonal entries $\{1,\ldots,1,\,1-l_l^{(l)}(\tau_m),\ldots,\,1-l_{k-2}^{(l)}(\tau_m)\}$ with

$$l_i^{(l)}(\tau_m):=\begin{cases} l_i^{(l)} & \text{if } \; l_i^{(l)}\neq 0 \\ \tau_m & \text{if } \; l_i^{(l)}=0 \end{cases}, \qquad i=l,\ldots,k-2.$$

Let us now define $\tilde{V}_l(\tau_m)$ $(l=0,\ldots,k-2)$ as the $k\times k$ upper bidiagonal stochastic matrix whose diagonal entries are $\{l_l^{(l)}(\tau_m),\ldots,\,l_{k-2}^{(l)}(\tau_m),\,1,\ldots,1\}$. Let $\tilde{V}(\tau_m):=\tilde{V}_{k-2}(\tau_m)\cdots\tilde{V}_0(\tau_m)$, $\tilde{L}(\tau_m):=\tilde{L}_{k-2}(\tau_m)\cdots\tilde{L}_0(\tau_m)$ and

$$V(\tau_m):=\begin{pmatrix} \tilde{V}(\tau_m) & 0 \\ 0 & I_{s+1-k} \end{pmatrix}, \qquad L(\tau_m):=\begin{pmatrix} I_{r+1-k} & 0 \\ 0 & \tilde{L}(\tau_m) \end{pmatrix}.$$

Let $(u_{0,m}, ..., u_{n,m})$ $(m \in \mathbf{N})$ be the system of functions defined on I such that

$$(u_{0,m}, ..., u_{r,m}) := (u_0, ..., u_r)\, L^{-1} L(\tau_m); \qquad u_{i,m} := u_i, \quad i = r+1, ..., n. \tag{6.13}$$

By construction $\lim_{m \to \infty} L(\tau_m) = L$ and therefore

$$\lim_{m \to \infty} (u_{0,m}(t), ..., u_{n,m}(t)) = (u_0(t), ..., u_n(t)), \qquad \forall t \in I. \tag{6.14}$$

Taking into account that L^{-1}, $L(\tau_m)$ are stochastic matrices and that $(u_0, ..., u_n)$ is normalized on I we deduce that $(u_{0,m}, ..., u_{n,m})$ $(m \in \mathbf{N})$ is also a normalized system on I. Since $L^{-1} L(\tau_m) = \begin{pmatrix} I_{r+1-k} & 0 \\ 0 & \tilde{L}^{-1} \tilde{L}(\tau_m) \end{pmatrix}$ then $u_{i,m} = u_i$ for all $i = 0, ..., r-k$. Moreover, by (2.11), $u_i(t) = u_{i,m}(t) = 0$, $\forall t \in I''$, for all $i = 0, ..., r-k$. Therefore $1 = \sum_{i=0}^{n} u_{i,m}(t) = \sum_{i=r-k+1}^{n} u_{i,m}(t)$, $\forall t \in I''$. Since $l_i^{(l)}(\tau_m) \neq 0$ $(l = 0, ..., k-2,\ i = l, ..., k-2)$ then $V(\tau_m)$ is nonsingular and we can define the system $(z_{0,m}, ..., z_{s,m}) := (u_{n-s,m}, ..., u_{n,m})\, V^{-1}(\tau_m)$ $(m \in \mathbf{N})$ and so

$$(z_{0,m}, ..., z_{k-1,m}) = (u_{n-s,m}, ..., u_{r,m})\, \tilde{V}^{-1}(\tau_m). \tag{6.15}$$

Taking into account (6.13) and that, by Remark 2.6, $n + k - s = r + 1$ we deduce that

$$z_{i,m} = u_{n-s+i,m} = u_{n-s+i}, \qquad i = k, ..., s. \tag{6.16}$$

Reasoning as we have done above with the system defined in (6.2) one can deduce that $(z_{0,m}, ..., z_{s,m})$ $(m \in \mathbf{N})$ is a normalized system on I'' and that

$$z_{i,m}^{(j)}(t_0) = 0, \quad j < i; \qquad z_{i,m}^{(i)}(t_0) > 0, \quad i = 0, ..., k-1. \tag{6.17}$$

Moreover, since $(u_0, ..., u_n)$ is formed by continuous functions at t_0 then $(u_{0,m}, ..., u_{n,m})$ and $(z_{0,m}, ..., z_{s,m})$ $(m \in \mathbf{N})$ are also formed by continuous functions at t_0 and, by (6.17) and Taylor formula, there exists $0 < \delta < \varepsilon$ such that $z_{i,m}(t) > 0$, $\forall t \in [t_0, t_0 + \delta)$, for all $i = 0, ..., k-1$. Since by (6.16) $z_{i,m} = u_{n-s+i}$ for all $i = k, ..., s$ and $(u_0, ..., u_n)$ is formed by nonnegative functions then $(z_{0,m}, ..., z_{s,m})$ is normalized and formed by nonnegative functions on $[t_0, t_0 + \delta)$. Therefore, given $t \in [t_0, t_0 + \delta)$, $(z_{i,m}(t))_{m \in \mathbf{N}}$ is contained in the compact set $[0, 1]$ and there exists $(\tau_{m_{j(t)}})_{j(t) \in \mathbf{N}} \subseteq (\tau_m)_{m \in \mathbf{N}}$ such that $\lim_{j(t) \to \infty} \tau_{m_{j(t)}} = 0$ and $(z_{i, m_{j(t)}}(t))_{j(t) \in \mathbf{N}}$ is a convergent sequence. Let us define

$$\tilde{z}_i(t) := \lim_{j(t) \to \infty} z_{i, m_{j(t)}}(t), \qquad t \in [t_0, t_0 + \delta)$$

for $i=0, ..., k-1$ and $\tilde{V}(0) := \lim_{m\to\infty} \tilde{V}(\tau_m)$. Since $\lim_{m\to\infty} l_{i_0}^{(l_0)}(\tau_m)=0$ then $\tilde{V}_{l_0}(0) := \lim_{m\to\infty} \tilde{V}_{l_0}(\tau_m)$ is a singular matrix and therefore $\tilde{V}(0)$ is singular. From (6.14) and (6.15) we can write

$$\begin{aligned}(u_{n-s}(t), ..., u_r(t)) &= \lim_{j\to\infty} (z_{0, m_{j(t)}}(t), ..., z_{k-1, m_{j(t)}}(t))\, \tilde{V}(\tau_{m_{j(t)}}) \\ &= (\tilde{z}_0(t), ..., \tilde{z}_{k-1}(t))\, \tilde{V}(0), \qquad t\in[t_0, t_0+\delta). \qquad (6.18)\end{aligned}$$

By (6.18) and the singularity of $\tilde{V}(0)$, $(u_{n-s}, ..., u_r)$ is a linearly dependent system of functions on $[t_0, t_0+\delta)$ and this contradicts that, by Lemma 2.13, $u_{n-s}, ..., u_r$ are linearly independent on any interval J such that $t_0 \in J \subseteq I$. ∎

We finish this section with some matricial aspects for symmetric B-algorithms.

Remark 6.2. Let us now observe that the matrix M of the B-algorithm corresponding to admissible parameters can be always factorized as a product of rectangular totally positive matrices and hence is also totally positive. A nonsingular totally positive matrix is called *almost strictly totally positive* (ASTP) if it satisfies that $\det A[\alpha, \beta] > 0$ if and only if its diagonal entries are all positive. Theorem 4.4 of [10] characterizes this class of matrices in terms of their bidiagonal factorizations. If the B-algorithm is symmetric then one can check (applying that result to factorizations (2.14) and (2.15)) that the matrices L and U corresponding to the left and right B-algorithm are both ASTP.

7. EXAMPLES

In this section we shall see many examples of spaces with symmetric B-algorithms. However, we start with an example showing that when hypotheses of Theorem 6.1 do not hold there exist B-algorithms which are not symmetric.

EXAMPLE 7.1. Let $\mathscr{U} = \operatorname{span}\{u_0(t), u_1(t)\}$, $t\in[0, 1]$,

$$u_0(t) := \begin{cases} 1-t & \text{if } 0\leqslant t\leqslant 1/4 \\ \dfrac{2-2t}{3} & \text{if } 1/4 < t\leqslant 1, \end{cases} \qquad u_1(t) := \begin{cases} t & \text{if } 0\leqslant t\leqslant 1/4 \\ \dfrac{1+2t}{3} & \text{if } 1/4 < t\leqslant 1. \end{cases}$$

It can be checked by using Proposition 2.3 that (u_0, u_1) form a (normalized) B-basis of $\mathscr{U}$. Let $t_0 = 1/4$ and let us consider the corresponding B-algorithm. Then the normalized B-basis (v_0, v_1, w_0, w_1) of $\mathscr{V}$ satisfies $(v_0(t), v_1(t)) = (1-4t, 4t)$, $t\in I' = [0, 1/4]$, $(w_0(t), w_1(t)) = (\frac{4-4t}{3}, \frac{4t-1}{3})$, $t\in I'' = (1/4, 1]$, and one has that the matrix L of (2.12) is $L=\begin{pmatrix} 1 & 0 \\ 3/4 & 1/4 \end{pmatrix}$ and the matrix U of (2.13) is $U=\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$. Therefore the B-algorithm is not symmetric. Let us observe that the basis functions do not vanish at $t_0=1/4$ and they are not continuous at this parameter, and so $t_0=1/4$ is not an admissible parameter.

The following example illustrates that continuity of the functions at a given parameter can be sufficient in order to get a symmetric B-algorithm (which provides a knot insertion algorithm).

EXAMPLE 7.2. Let $\mathscr{U}=\text{span}\{u_0(t), u_1(t), u_2(t)\}$, $t\in[0, 2]$, $I' := [0, 1]$, $I''=(1, 2]$.

$$u_0(t) := \begin{cases} 1-t^2, & t\in I' \\ 0, & t\in I'', \end{cases}$$

$$u_1(t) := \begin{cases} t^2, & t\in I' \\ (2-t)^2, & t\in I'', \end{cases}$$

$$u_2(t) := \begin{cases} 0, & t\in I' \\ 1-(2-t)^2, & t\in I''. \end{cases}$$

It can be checked by using Proposition 2.3 that (u_0, u_1, u_2) form a (normalized) B-basis of $\mathscr{U}$. The number of basis functions which do not vanish at $t_0=1$ is 1 and this is a 1-admissible parameter. Therefore, by Theorem 6.1, the B-algorithm is symmetric. In fact the matrix M of the B-algorithm (see (2.10)) is

$$M=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the following example we apply the knot insertion algorithm obtained with the procedure given in Theorem 5.1 in the space of polynomial splines and we obtain the well known de Boor–Cox algorithm.

EXAMPLE 7.3. Let $\mathbf{x} := \{a=x_0=\cdots=x_{n-1}<x_n\leqslant\cdots\leqslant x_{\ell+n-2}< x_{\ell+n-1}=\cdots=x_{\ell+2n-2}=b\}\subset\mathbf{R}$, be an ordered sequence of real values satisfying $0<x_{n+i}-x_i$ for all $i=0, ..., \ell+n-2$. Let $\mathscr{S}^n_{\mathbf{x}}$ be the space of spline functions of degree n defined on $I=[a, b]$ with knot vector $\mathbf{x}$. The

basis functions of this space can be defined recursively by the Mansfield–de Boor–Cox recursion (see [9, Chap. 10])

$$N_i^0(x)=\begin{cases}1, & \text{if} \quad x_{i-1}\leqslant x<x_i\\ 0, & \text{elsewhere}\end{cases};$$

$$N_i^k(x)=\frac{x-x_{i-1}}{x_{i+k-1}-x_{i-1}}N_i^{k-1}(x)+\frac{x-x_i}{x_{i+k}-x_i}N_{i+1}^{k-1}(x)$$

for $k=1, ..., n$ and then $\operatorname{supp}(N_i^n)=[x_{i-1}, x_{i+n}]$. In Theorem 4.6 of [5] it was proved that $(N_1^n, ..., N_{\ell+n-1}^n)$ is the normalized B-basis of $\mathscr{S}_{\mathbf{x}}^n$. Let us suppose that $x_j\leqslant t_0<x_{j+1}$ and $0\leqslant p\leqslant n$ is the multiplicity of t_0 as knot of $\mathbf{x}$ (i.e., $x_{j-p}<x_{j-p+1}=\cdots=x_j=t_0$). By considering the support of the basis functions one can deduce that the dimension of $\mathscr{S}_{\mathbf{x}}^n|_{I'}$ (resp., $\mathscr{S}_{\mathbf{x}}^n|_{I''}$) is $r+1:=j-p+1$ (resp., $s+1:=\ell+2n-j-1$) and hence the potential knot multiplicity of t_0 is $k:=n+1-p$ which coincides with the number of basis functions that do not vanish at t_0. The normalized B-basis of $\mathscr{S}_{\mathbf{x}}^n|_{I'}$ is the B-spline basis of degree n and knot vector $\mathbf{x}':=\{(a=)\,x_0=\cdots=x_{n-1}<x_n\leqslant\cdots\leqslant x_{j-p}<x_{j-p+1}=\cdots=x_j=t_0=\cdots=t_0\}$ and the normalized B-basis of $\mathscr{S}_{\mathbf{x}}^n|_{I''}$ is the B-spline basis of degree n and knot vector $\mathbf{x}'':=\{x_{j-p+1}=\cdots=x_j=t_0=\cdots=t_0<x_{j+1}\leqslant\cdots\leqslant x_{\ell+n-2}<x_{\ell+n-1}=\cdots=x_{\ell+2n-2}\}$. Let $L=L_{k-2}\cdots L_0$ and $U=U_{k-2}\cdots U_0$ be the factorizations (2.14) and (2.15) of the matrices satisfying (2.12) and (2.13), respectively. The diagonal entries $1-l_i^{(j)}$ $(0\leqslant i\leqslant r)$ and $u_i^{(j)}$ $(0\leqslant i\leqslant s)$ of L_j and U_j, respectively, satisfy $1-l_i^{(j)}=1$ for $i=0, ..., j-n+j$, $u_i^{(j)}=1$ for $i=n-p-j, ..., \ell+2n-j-2$ and $l_i^{(j)}=(t_0-x_i)/(x_{i+n-j}-x_i)=u_{i-(j-n+1+j)}^{(j)}$ for $i=j-n+j+1, ..., j-p$. The corresponding B-algorithm is so symmetric and the knot insertion algorithm constructed in Theorem 5.1 coincides with the well known de Boor–Cox algorithm.

As we are now going to illustrate in the particular example of the space of polynomials, we can always apply our techniques to any space of functions with symmetric B-algorithms and obtain new spaces associated to knot vectors.

EXAMPLE 7.4. If $\mathscr{U}=\mathscr{P}^n([a,b])$ is the space of polynomials of degree less than or equal to n defined on $[a,b]$, we know that the normalized B-basis $(u_0, ..., u_n)$ of $\mathscr{U}$ is the Bernstein basis $u_i(t):=\binom{n}{i}\left(\frac{t-a}{b-a}\right)^i\left(\frac{b-t}{b-a}\right)^{n-i}$, $i=0, ..., n$. Given $t_0\in(a,b)$, then the normalized B-bases $(\bar{v}_0, ..., \bar{v}_n)$ and $(\bar{w}_0, ..., \bar{w}_n)$ of $\mathscr{U}|_{I'}$ and $\mathscr{U}|_{I''}$, respectively, are given by $\bar{v}_i(t):=\binom{n}{i}\left(\frac{t-a}{t_0-a}\right)^i\left(\frac{t_0-t}{t_0-a}\right)^{n-i}$, $t\in I'$; $\bar{w}_i(t):=\binom{n}{i}\left(\frac{t-t_0}{b-t_0}\right)^i\left(\frac{b-t}{b-t_0}\right)^{n-i}$, $t\in I''$, $i=0, ..., n$. The potential knot multiplicity of t_0 in $\mathscr{U}$ is $k:=n+1$. Let $L=L_{n-1}\cdots L_0$ and $U=U_{n-1}\cdots U_0$ be the factorizations (2.14) and (2.15) of the matrices satisfying (2.12) and (2.13), respectively. If $1-l_\ell^{(j)}$ and $u_{\ell-(j+1)}^{(j)}$ are the diagonal entries of L_j and U_j, respectively then $l_\ell^{(j)} = \frac{b-t_0}{b-a} = u_{\ell-(j+1)}^{(j)}$ for all $l = j+1, ..., n$ and the corresponding B-algorithm is symmetric. The system $(v_0, ..., v_n, w_0, ..., w_n)$ with v_i and w_j defined in (2.5) and (2.6) coincides with the B-spline basis of degree n defined on $[a, b]$ and knot vector $\mathbf{x}_{n+1} := \{x_0 = \cdots = x_n < x_{n+1} = \cdots = x_{2n+1} < x_{2n+2} = \cdots = x_{3n+2}\}$ with $x_i = a$, $i = 0, ..., n$, $x_i = t_0$, $i = n+1, ..., 2n+1$ and $x_i = b$, $i = 2n+2, ..., 3n+2$. Its generated space is $\mathscr{V} = \mathscr{S}^n_{\mathbf{x}_{n+1}}([a, b])$. Our B-algorithm provides a knot insertion algorithm and let us observe that the system defined in (4.2) is the B-spline basis of degree n and knot vector $\mathbf{x}_p := \{x_0 = \cdots = x_n < x_{n+1} = \cdots = x_{n+p} < x_{n+p+1} = \cdots = x_{2n+1+p}\}$ with $x_i = a$, $i = 0, ..., n$, $x_i = t_0$, $i = n+1, ..., n+p$ and $x_i = b$, $i = n+p+1, ..., 2n+p+1$. The generated space is $\mathscr{U}^{n+p+1} = \mathscr{S}^n_{\mathbf{x}_p}([a, b])$.

We now present extended Tchebycheff spaces and Tchebycheffian spline spaces as particular examples where we can apply our techniques to obtain knot insertion algorithms.

EXAMPLE 7.5. A system of functions $(u_0, ..., u_n)$ in $C^n(I)$ and its generated space $\mathscr{U}$ are called *extended Tchebycheff* if for any $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ $\det(u_j^{(m_i)}(t_i))_{0 \leqslant i, j \leqslant n} > 0$ $(m_i = \#\{k < i \mid t_k = t_i\})$. Let $\Delta := \{a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b\}$ be a partition of $I = [a, b]$ and let $\mathscr{M} := \{m_1 < \cdots < m_k\}$ be a vector of integers with $1 \leqslant m_i \leqslant n+1$ for all $i = 1, ..., k$. The space of *Tchebycheffian spline functions* with knots $x_1, ..., x_k$ of multiplicities $m_1, ..., m_k$ is defined in Chapter 9 of [19] as

$$\mathscr{S}(\mathscr{U}, \mathscr{M}, \Delta) = \left\{ \begin{array}{ll} s\colon I \to \mathbf{R} \mid s_i := s|_{(x_i, x_{i+1})} \in \mathscr{U}, & i = 0, ..., k \\ s_{i-1}^{(j-1)}(x_i) = s_i^{(j-1)}(x_i), & j = 1, ..., n+1-m_i;\ i = 1, ..., k \end{array} \right\}.$$

Polynomial, exponential and hyperbolic B-splines are included in the class of Tchebycheffian splines. In Theorem 5.1 of [6] it was proved that an extended Tchebycheff space of functions defined on a compact interval $I = [a, b]$ has always a totally positive basis and hence B-bases. From the definition it follows that the Hermite interpolation problem on an extended Tchebycheff space always has a (unique) solution. The number of zeros (counted with their multiplicity) of a nonzero function in an $(n+1)$-dimensional extended Tchebycheff space is less than or equal to n. Taking into account the previous facts one can easily deduce from Propositions 2.1(i) and 2.3 that if $(b_0, ..., b_n)$ is a B-basis of an extended Tchebycheff space $\mathscr{U}$ of functions defined on $[a, b]$ then

$$b_i^{(j)}(a) = 0, \quad j = 0, ..., i-1, \qquad b_i^{(j)}(b) = 0, \quad j = 0, ..., n-i-1$$

$$b_i^{(i)}(a) \neq 0, \qquad b_i^{(n-i)}(b) \neq 0.$$

Given an extended Tchebycheff space $\mathscr{U}$ of functions defined on $[a, b]$, properties (A) and (B) of Definition 2.9 are clearly satisfied by all parameters in (a, b) and so they are admissible. Then, given any extended Tchebycheff space with a normalized totally positive basis by Theorem 6.1, the corresponding B-algorithms are symmetric and, by Theorem 5.1, they provide a knot insertion algorithm. We can iterate this procedure with other parameters. In this process, we transform extended Tchebycheff spaces into extended Tchebycheffian spline spaces and, by construction, also obtain the normalized B-bases of the obtained spaces.

EXAMPLE 7.6. In [12] there are classes of "generalized B-spline" functions which includes Tchebycheffian B-splines and certain trigonometric B-splines. These "generalized B-splines" form a locally supported totally positive basis (see Section 1 and Theorem 4 of [12]). It can be checked, by using property (B) of Section 1 of [12] joined with Propositions 2.1(i) and 2.3, that it is a B-basis. Let us see that if the generated space has a *normalized* totally positive basis then a knot insertion algorithm can be derived. Property (A) of Definition 2.9 is clearly satisfied by interior parameters. From Theorem 3 of [12] we can deduce the existence of functions f_i whose derivatives satisfy $f_i^{(j)} = \delta_{ij}$ $(0 \leqslant j \leqslant i)$ and, by Remark 2.10, property (B) of Definition 2.9 also holds, and so interior parameters are all admissible. Then, by Theorem 6.1 the corresponding B-algorithms are symmetric and, by Theorem 5.1, they provide a knot insertion algorithm.

REFERENCES

1. P. J. Barry, de Boor–Fix functionals and algorithms for Tchebycheffian B-spline curves, *Constr. Approx.* **12** (1996), 385–408.
2. D. Bister, "Ein neuer Zugang für eine verallgemeinerte Klasse von Tschebyscheff Splines," Dissertation, Universität Karlsruhe, Shaker Verlag, Aachen, 1997.
3. D. Bister and H. Praitzsch, A new approach to Tchebycheffian B-splines, *in* "Curves and Surfaces with Applications in CAGD" (A. Le Méhauté, C. Rabut, and L. L. Schumaker, Eds.), pp. 35–41, Peters, Boston, 1997.
4. J. M. Carnicer and J. M. Peña, Shape preserving representations and optimality of the Bernstein basis, *Adv. Comput. Math.* **1** (1993), 173–196.
5. J. M. Carnicer and J. M. Peña, Totally positive bases for shape preserving curve design and optimality of B-splines, *Comput. Aided Geom. Design* **11** (1994), 635–656.
6. J. M. Carnicer and J. M. Peña, On transforming a Tchebycheff system into a strictly totally positive system, *J. Approx. Theory* **81** (1995), 274–295.
7. J. M. Carnicer, J. M. Peña, and R. Zalik, On strictly totally positive systems, *J. Approx. Theory* **98** (1998), 411–441.
8. N. Dyn and A. Ron, Recurrence relations for Tchebycheffian B-splines, *J. Anal. Math.* **51** (1988), 118–138.
9. G. Farin, "Curves and Surfaces for Computer Aided Geometric Design," 4th ed., Academic Press, San Diego, 1996.

10. M. Gasca and J. M. Peña, On factorization of totally positive matrices, *in* “Total Positivity and Its Applications” (M. Gasca and C. A. Micchelli, Eds.), pp. 109–130, Kluwer Academic, Dordrecht, 1996.
11. T. N. T. Goodman, Shape preserving representations, *in* “Mathematical Methods in CAGD” (T. Lyche and L. L. Schumaker, Eds.), pp. 333–357, Academic Press, Boston, 1989.
12. T. N. T. Goodman and S. L. Lee, Interpolatory and variation-diminishing properties of generalized B-splines, *Proc. Roy. Soc. Edinburgh Sect. A* **96** (1984), 249–259.
13. S. Karlin, “Total Positivity,” Stanford Univ. Press, Stanford, 1968.
14. T. Lyche, A recurrence relation for Chebyshevian B-splines, *Constr. Approx.* **1** (1985), 155–163.
15. T. Lyche and L. L. Schumaker, Total positivity properties of LB-splines, *in* “Total Positivity and Its Applications” (M. Gasca and C. A. Micchelli, Eds.), pp. 35–46, Kluwer Academic, Dordrecht, 1996.
16. E. Mainar and J. M. Peña, Corner cutting algorithms associated with optimal shape preserving representations, *Comput. Aided Geom. Design* **16** (1999), 883–906.
17. M. L. Mazure and H. Pottman, Tchebycheff curves, *in* “Curves and Surfaces in Geometric Design” (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), pp. 187–218, Peters, Boston, 1996.
18. H. Pottman, The geometry of Tchebycheffian splines, *Comput. Aided Geom. Design* **10** (1993), 181–210.
19. L. L. Schumaker, “Spline Functions: Basic Theory,” Wiley, New York, 1981.
20. M. Sommer and H. Strauss, Weak Descartes systems in generalized spline spaces, *Constr. Approx.* **4** (1988), 133–145.